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6. Term rewrite systems

/ Term rewrite systems

6.1. Matchings

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. If there exists an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\mathfrak{t}' = \overline{\sigma} \cdot \mathfrak{t}$, then

- $\mathfrak{t}$ is a *prefix* of $\mathfrak{t}'$. This property is denoted by $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$;
- the $\mathcal{S}, \mathsf{V}$-substitution σ is a *matching* of $\mathfrak{t}$ into $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t}' := \mathsf{c}_3 \mathsf{c}_0 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_3 \rfloor \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor.$$

- Let $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3$. Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_3), (\mathsf{v}_3, \mathsf{c}_1 \mathsf{v}_1)]$. Since $\mathfrak{t}' = \overline{\sigma} \cdot \mathfrak{t}$, σ is a matching of $\mathfrak{t}$ into $\mathfrak{t}'$ and $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$.
- Let $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor$. Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_3), (\mathsf{v}_3, \mathsf{v}_1)]$. Since $\mathfrak{t}' = \overline{\sigma} \cdot \mathfrak{t}$, σ is a matching of $\mathfrak{t}$ into $\mathfrak{t}'$ and $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$.
- Let $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_2$. By assuming that a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution σ exists such that σ is a matching of $\mathfrak{t}$ into $\mathfrak{t}'$, we would have both $\sigma \cdot \mathsf{v}_2 = \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_3$ and $\sigma \cdot \mathsf{v}_2 = \mathsf{c}_1 \mathsf{v}_1$. This is absurd, so that $\mathfrak{t}$ is not a prefix of $\mathfrak{t}'$.
- Let $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2$. By assuming that a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution σ exists, we would have $\mathfrak{t}' = \overline{\sigma} \cdot \mathfrak{t}$, so that $\mathsf{c}_3 = \mathfrak{t}' \cdot \epsilon = \overline{\sigma} \cdot \mathfrak{t} \cdot \epsilon = \mathsf{c}_2$. This is absurd, so that $\mathfrak{t}$ is not a prefix of $\mathfrak{t}'$.

Exercise ●○○○○

Consider the two $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms $\mathfrak{t}' := \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3 \rfloor$ and $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3 \rfloor$. Give two different matchings of $\mathfrak{t}$ into $\mathfrak{t}'$.

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term $\mathfrak{t}' := \mathsf{c}_3 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_2 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor \mathsf{c}_0 \rfloor$. Give examples of $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ are prefixes of $\mathfrak{t}'$ of constant lengths n for all $n \in [\![5]\!]$.

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term $\mathfrak{t} := \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_1 \rfloor$. Give an example of a ground $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term $\mathfrak{t}'$ such that $\ell_{\mathrm{cns}} \cdot \mathfrak{t}' = 9$ and $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$.

Exercise ●○○○○

Define two different $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ satisfying $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$ and $\mathfrak{t}' \preccurlyeq_{\mathrm{p}} \mathfrak{t}$.

A binary relation $\mathcal{R}$ on a set X is a *preorder* if $\mathcal{R}$ is reflexive and transitive.

When, additionally, $\mathcal{R}$ is antisymmetric, $\mathcal{R}$ is an *order relation*.

Proposition [Prefix relation on $\mathcal{S}, \mathrm{V}$-terms]

For any signature $\mathcal{S}$ and set of variables V, the binary relation $\preccurlyeq_{\mathrm{p}}$ is

- ☐ a preorder, when seen as a binary relation on $\mathcal{S}, \mathrm{V}$-terms;
- ☐ an order relation, when seen as a binary relation on planar labeled $\mathcal{S}$-terms.

Exercise ●○○○○

Give a simple combinatorial characterization of the fact that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$ holds, where $\mathfrak{t}$ and $\mathfrak{t}'$ are two planar labeled $\mathcal{S}$-terms and $\mathcal{S}$ is a signature.

Exercise ●●○○○

Prove the previous proposition.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms.

A matching of $\mathfrak{t}$ into $\mathfrak{t}'$ can be computed through the ARS $\mathsf{Matching}_{\mathcal{S},\mathsf{V}} := \left(\{\mathsf{Fail}\} \cup \mathcal{P} \cdot \underline{\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}}^2, \Rightarrow\right)$ such that

$$\{(c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, c\, \mathfrak{t}'_1 \dots \mathfrak{t}'_n)\} \sqcup S \Rightarrow \{(\mathfrak{t}_1, \mathfrak{t}'_1), \dots, (\mathfrak{t}_n, \mathfrak{t}'_n)\} \cup S,$$

$$\{(c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, c'\, \mathfrak{t}'_1 \dots \mathfrak{t}'_{n'})\} \sqcup S \Rightarrow \mathsf{Fail}, \quad \text{if } c \neq c',$$

$$\{(c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, v)\} \sqcup S \Rightarrow \mathsf{Fail},$$

$$\{(v, \mathfrak{t}), (v, \mathfrak{t}')\} \sqcup S \Rightarrow \mathsf{Fail}, \quad \text{if } \mathfrak{t} \neq \mathfrak{t}'.$$

This ARS is used by computing the normal form of $\{(\mathfrak{t}, \mathfrak{t}')\}$ and, when this normal form is a set S, by considering the $\mathcal{S}, \mathsf{V}$-substitution $[S]$ specified by S.

This $\mathcal{S}, \mathsf{V}$-substitution satisfies $\mathfrak{t}' = \overline{[S]} \cdot \mathfrak{t}$.

The normal form Fail is reached in the case where $\mathfrak{t}$ is not a prefix of $\mathfrak{t}'$.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_3 v_1 \llcorner c_2 v_2 v_3 \lrcorner v_1$ and $t' := c_3 \llcorner c_1 v_2 \lrcorner \llcorner c_2 \llcorner c_1 v_2 \lrcorner v_3 \lrcorner \llcorner c_1 v_2 \lrcorner$. In $\mathsf{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_1 v_2), (c_2 v_2 v_3, c_2 \llcorner c_1 v_2 \lrcorner v_3), (v_1, c_1 v_2)\} \Rightarrow \{(v_1, c_1 v_2), (v_2, c_1 v_2), (v_3, v_3), (v_1, c_1 v_2)\}.$$

We can check that the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-substitution $[\{(v_1, c_1 v_2), (v_2, c_1 v_2), (v_3, v_3)\}]$ is a matching of t into t'.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_2 v_1 \llcorner c_2 v_2 v_3 \lrcorner$ and $t' := c_2 c_0 \llcorner c_3 v_1 v_2 v_3 \lrcorner$. In $\mathsf{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_0), (c_2 v_2 v_3, c_3 v_1 v_2, v_3)\} \Rightarrow \mathsf{Fail}.$$

We can check that t is not a prefix of t'.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_3 v_1 v_2 v_2$ and $t' := c_3 c_0 \llcorner c_1 c_0 \lrcorner \llcorner c_1 \llcorner c_1 c_0 \lrcorner \lrcorner$. In $\mathsf{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_0), (v_2, c_1 c_0), (v_2, c_1 \llcorner c_1 c_0 \lrcorner)\} \Rightarrow \mathsf{Fail}.$$

We can check that t is not a prefix of t'.

Due to the following results, the ARS $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$, where $\mathcal{S}$ is any signature and V is any set of variables, computes exactly what is described before.

Theorem [Convergence of $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$]

For any signature $\mathcal{S}$ and set of variables V, the ARS $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$ is convergent.

Proposition [Computation of $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, and $S := \{(\mathfrak{t}, \mathfrak{t}')\}$.

- If $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$, then the normal form of S in $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$ is the set S' such that $[S']$ is a matching of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Otherwise, the normal form of S in $\mathsf{Matching}_{\mathcal{S},\mathrm{V}}$ is Fail.

6.2. Rewrite mechanics

Definition

A *term rewrite system* (*TRS*) is a triple $(\mathcal{S}, \mathrm{V}, \to)$ where $\mathcal{S}$ is a signature, called the *underlying signature*, V is a set of variables, called the *underlying set of variables*, and $\to$ is a binary relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, called the *elementary rewrite relation*.

The elementary rewrite relation must satisfy the following two conditions:

(T1) for any $\mathcal{S}, \mathrm{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \to \mathfrak{t}'$ implies that $\mathrm{Vars} \cdot \mathfrak{t}' \subseteq \mathrm{Vars} \cdot \mathfrak{t}$;

(T2) for any $\mathcal{S}, \mathrm{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \to \mathfrak{t}'$ implies that $\ell_{\mathrm{cns}} \cdot \mathfrak{t} \geqslant 1$.

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS.

If $\mathfrak{t}$ and $\mathfrak{t}'$ are two $\mathcal{S}, \mathrm{V}$-terms such that $\mathfrak{t} \to \mathfrak{t}'$, then

- the pair $(\mathfrak{t}, \mathfrak{t}')$ is a *rewrite rule* of $\mathcal{T}$;
- $\mathfrak{t}$ is the *left-hand side* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$;
- $\mathfrak{t}'$ is the *right-hand side* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$;
- $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t} \cdot 1$ is the *head constant* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$. By (T2), this is well-defined;
- for any $\mathcal{S}, \mathrm{V}$-substitution σ, the pair $(\overline{\sigma} \cdot \mathfrak{t}, \overline{\sigma} \cdot \mathfrak{t}')$ is an *instance* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$.

Example

Let $\mathsf{Assoc} := ((\{m\}, \mathrm{ar}), \{1, 2, 3\}, \rightarrow)$ be the TRS such that $\mathrm{ar}{\cdot}m = 2$ and $m \lfloor \underline{m12} \rfloor 3 \rightarrow m1 \lfloor \underline{m23} \rfloor$.

Example

Let $\mathsf{Assoc}_2 := ((\{m, m'\}, \mathrm{ar}), \{1, 2, 3\}, \rightarrow)$ be the TRS such that $\mathrm{ar}{\cdot}m = 2$, $\mathrm{ar}{\cdot}m' = 2$, $m \lfloor \underline{m12} \rfloor 3 \rightarrow m1 \lfloor \underline{m23} \rfloor$, and $m'1 \lfloor \underline{m'23} \rfloor \rightarrow m' \lfloor \underline{m'12} \rfloor 3$.

Example

Let $\mathsf{Eq} := ((\{t, e\}, \mathrm{ar}), \{1\}, \rightarrow)$ be the TRS such that $\mathrm{ar}{\cdot}t = 0$, $\mathrm{ar}{\cdot}e = 2$, and $e11 \rightarrow t$.

Example

Let $\mathsf{NatAdd} := ((\{z, s, a\}, \mathrm{ar}), \{1, 2\}, \rightarrow)$ be the TRS such that $\mathrm{ar}{\cdot}z = 0$, $\mathrm{ar}{\cdot}s = 1$, $\mathrm{ar}{\cdot}a = 2$, $a1z \rightarrow 1$, and $a1 \lfloor \underline{s2} \rfloor \rightarrow s \lfloor \underline{a12} \rfloor$.

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \rightarrow)$ be a TRS.

The *rewrite relation* of $\mathcal{T}$ is the smallest binary relation $\Rightarrow$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$ satisfying the two following properties:

(1) for any instance $(\mathfrak{s}, \mathfrak{s}')$ of a rewrite rule of $\mathcal{T}$,

$$\mathfrak{s} \Rightarrow \mathfrak{s}';$$

(2) for any $c \in \mathcal{S} \cdot n$, $n \geqslant 1$, and any $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, $i \in [n]$, such that $\mathfrak{t}_i \Rightarrow \mathfrak{t}'_i$,

$$c\, \mathfrak{t}_1 \dots \mathfrak{t}_i \dots \mathfrak{t}_n \Rightarrow c\, \mathfrak{t}_1 \dots \mathfrak{t}'_i \dots \mathfrak{t}_n.$$

The TRS $\mathcal{T}$ defines the ARS $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}, \Rightarrow)$, called *the ARS of* $\mathcal{T}$.

We use the previous notations in the context of ARSs, here on the ARS of $\mathcal{T}$. For instance, $\mathfrak{t} \mapsto \mathfrak{t}^{\Rightarrow^*}$ is the future function of $\mathcal{T}$ and $\equiv$ is the convertibility relation of $\mathcal{T}$.

Similarly, we write that $\mathcal{T}$ satisfies a property P for the fact that the ARS of $\mathcal{T}$ satisfies P. For instance, writing that $\mathcal{T}$ is confluent means that the ARS of $\mathcal{T}$ is confluent.

Example

In Assoc_2, by setting $\mathtt{t} := m' \lfloor m \lfloor m12 \rfloor 3 \rfloor \lfloor m' 4 \lfloor m' 56 \rfloor \rfloor$, we have

- ☐ $\mathtt{t} \Rightarrow m' \lfloor m' \lfloor m \lfloor m12 \rfloor 3 \rfloor 4 \rfloor \lfloor m' 56 \rfloor$;
- ☐ $\mathtt{t} \Rightarrow m' \lfloor m \lfloor m12 \rfloor 3 \rfloor \lfloor m' \lfloor m' 45 \rfloor 6 \rfloor$;
- ☐ $\mathtt{t} \Rightarrow m' \lfloor m1 \lfloor m23 \rfloor \rfloor \lfloor m' 4 \lfloor m' 56 \rfloor \rfloor$.

Exercise ●○○○○

Give a normalizing rewrite sequence in NatAdd starting from $a \lfloor sz \rfloor \lfloor s \lfloor sz \rfloor \rfloor$.

Exercise ●●○○○

Draw the rewrite graph of the closed sub-ARS of the ARS of Assoc generated by $\left\{ m \lfloor m \lfloor m \lfloor m12 \rfloor 3 \rfloor 4 \rfloor 5 \right\}$.

Exercise ●●○○○

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS. Prove that for any $\mathcal{S}, \mathsf{V}$-term $\mathtt{t}$, for any $\mathtt{t}' \in \mathtt{t}^{\Rightarrow^*}$, $\mathrm{Vars} \cdot \mathtt{t}' \subseteq \mathrm{Vars} \cdot \mathtt{t}$. From this property, prove that all $\mathcal{S}, \mathsf{V}$-terms of the future of any ground $\mathcal{S}, \mathsf{V}$-term are ground.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. If there exists a position w within $\mathfrak{t}'$ such that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}' \cdot w$, then

- $\mathfrak{t}$ is a *factor* of $\mathfrak{t}'$. This property is denoted by $\mathfrak{t} \preccurlyeq_{\mathrm{f}} \mathfrak{t}'$;
- the word w is an *occurrence of* $\mathfrak{t}$ *into* $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t}' := \mathsf{c}_2 \mathsf{v}_3 \underbracket{\mathsf{c}_3 \underbracket{\mathsf{c}_1 \mathsf{v}_2} \underbracket{\mathsf{c}_2 \mathsf{v}_3 \underbracket{\mathsf{c}_1 \mathsf{c}_0}} \underbracket{\mathsf{c}_1 \mathsf{v}_2}}.$$

- Let $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \underbracket{\mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3} \mathsf{v}_1$. Since $\mathfrak{t}$ is a prefix of $\mathfrak{t}' \cdot 2$, $\mathfrak{t}$ is a factor of $\mathfrak{t}'$ and 2 is an occurrence of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Let $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2$. Since $\mathfrak{t}$ is a prefix of $\mathfrak{t}'$ and of $\mathfrak{t}' \cdot 22$, ϵ and 22 are two occurrences of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Let $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1$. There is no position w within $\mathfrak{t}'$ such that $\mathfrak{t}$ is a prefix of $\mathfrak{t}' \cdot w$. Therefore, $\mathfrak{t}$ is not a factor of $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

A *holed* $\mathcal{S}, \mathrm{V}$-term is an $\mathcal{S}, \mathrm{V} \sqcup \{\square\}$-term $\mathfrak{t}$ such that $\ell_{\square} \cdot \mathfrak{t} = 1$ and $\square$ is a variable which does not belong to V. The *hole position* of $\mathfrak{t}$ is the position within $\mathfrak{t}$ of its unique leaf decorated by $\square$.

For any holed $\mathcal{S}, \mathrm{V}$-term $\mathfrak{s}$, any $\mathcal{S}, \mathrm{V}$-term $\mathfrak{t}$, and any $\mathcal{S}, \mathrm{V}$-substitution σ, let

$$\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma := \mathfrak{s} \curvearrowleft_{\square} \lfloor \bar{\sigma} \cdot \mathfrak{t} \rfloor.$$

Proposition [Factors of $\mathcal{S}, \mathrm{V}$-terms]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathrm{V}$-terms, and w be a position within $\mathfrak{t}'$. The two following assertions are equivalent:

(1) w is an occurrence of $\mathfrak{t}$ into $\mathfrak{t}'$;

(2) there exists a holed $\mathcal{S}, \mathrm{V}$-term $\mathfrak{s}$ having w as hole position and an $\mathcal{S}, \mathrm{V}$-substitution σ such that $\mathfrak{t}' := \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma$.

Example

By considering the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}$-terms $\mathfrak{t}'$ and $\mathfrak{t}$ of the previous first example, we have

$$\mathfrak{t}' = \blacktriangle \cdot \lfloor \mathsf{c}_2 \mathsf{v}_3 \square \rfloor \cdot \mathfrak{t} \cdot [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_2, \mathsf{v}_3), (\mathsf{v}_3, \mathsf{c}_1 \mathsf{c}_0)\}].$$

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term $\mathfrak{t}' := \mathsf{c}_3 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_2 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor \mathsf{c}_0 \rfloor$. Give examples of $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ are factors of $\mathfrak{t}'$ of constant lengths n for all $n \in [\![5]\!]$.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. For any $\mathcal{S}, \mathrm{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, show that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$ implies $\mathfrak{t} \preccurlyeq_{\mathrm{f}} \mathfrak{t}'$. Show that the converse of this property is false.

Exercise ●○○○○

Show that the set of factors of an $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term is infinite.

Exercise ●●●○○

For any signature $\mathcal{S}$ and any labeled $\mathcal{S}$-term $\mathfrak{t}'$, prove that the set of planar labeled $\mathcal{S}$-signature terms $\mathfrak{t}$ such that $\mathfrak{t}$ is a factor of $\mathfrak{t}'$ is finite.

Proposition [Rewrite relation of a TRS]

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. The rewrite relation $\Rightarrow$ of $\mathcal{T}$ is characterized by the fact that

$$\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s} \cdot \sigma \;\Rightarrow\; \blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s}' \cdot \sigma$$

where

(1) $\mathfrak{t}$ is a holed $\mathcal{S}, \mathrm{V}$-term;

(2) $\mathfrak{s}$ and $\mathfrak{s}'$ are two $\mathcal{S}, \mathrm{V}$-terms such that $\mathfrak{s} \to \mathfrak{s}'$;

(3) σ is an $\mathcal{S}, \mathrm{V}$-substitution.

Let us consider the notations used in the statement of the previous proposition.

The hole position w of $\mathfrak{t}$ is the *rewrite position* of the one-step rewrite from $\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s} \cdot \sigma$ to $\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s}' \cdot \sigma$ in $\mathcal{T}$. Note that this position w is also a position within $\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s} \cdot \sigma$ and $\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s}' \cdot \sigma$.

This property is written as $\blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s} \cdot \sigma \Rightarrow_w \blacktriangle \cdot \mathfrak{t} \cdot \mathfrak{s}' \cdot \sigma$.

The binary relation $\Rightarrow_\epsilon$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$ is the *one-step root rewrite relation* of $\mathcal{T}$.

Example

In NatAdd, we have

$$a\llcorner az\llcorner s\llcorner sz\lrcorner\lrcorner\lrcorner\llcorner s1\lrcorner \Rightarrow a\llcorner s\llcorner az\llcorner sz\lrcorner\lrcorner\lrcorner\llcorner s1\lrcorner$$

because

$$a\llcorner az\llcorner s\llcorner sz\lrcorner\lrcorner\lrcorner\llcorner s1\lrcorner = \blacktriangle\cdot a\square\llcorner s1\lrcorner\cdot\llcorner a1\llcorner s2\lrcorner\lrcorner\cdot[z, sz] \quad\Rightarrow\quad \blacktriangle\cdot a\square\llcorner s1\lrcorner\cdot\llcorner s\llcorner a12\lrcorner\lrcorner\cdot[z, sz] = a\llcorner s\llcorner az\llcorner sz\lrcorner\lrcorner\lrcorner\llcorner s1\lrcorner.$$

Example

In Eq, we have

$$e\llcorner et3\lrcorner\llcorner et3\lrcorner \Rightarrow t$$

because

$$e\llcorner et3\lrcorner\llcorner et3\lrcorner = \blacktriangle\cdot\square\cdot\llcorner e11\lrcorner\cdot[et3] \quad\Rightarrow\quad \blacktriangle\cdot\square\cdot t\cdot[et3] = t.$$

Let TRS $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \rightarrow)$ be a TRS.

Given $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$, there can exist different positions w_1 and w_2 within $\mathfrak{t}$ such that $w_1 \neq w_2$, $\mathfrak{t} \Rightarrow_{w_1} \mathfrak{t}'$, and $\mathfrak{t} \Rightarrow_{w_2} \mathfrak{t}'$.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}, \rightarrow)$ such that $\mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \rightarrow \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2$ and $\mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \rightarrow \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3$.

We have $\mathsf{c}_2 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \rfloor \mathsf{v}_3 \Rightarrow_{\epsilon} \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3$ and $\mathsf{c}_2 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \rfloor \mathsf{v}_3 \Rightarrow_1 \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3$.

In order to keep track of these multiplicities, let $\mathrm{f}_{\Rightarrow} \cdot \mathfrak{t}$ be the $\mathbb{K}, \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$-series, where $\mathbb{K}$ is any field, defined by

$$\mathrm{f}_{\Rightarrow} \cdot \mathfrak{t} := \sum_{w \in \mathrm{P} \cdot \mathfrak{t}} \sum_{\mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}} [\mathfrak{t} \Rightarrow_w \mathfrak{t}'] \, \mathfrak{t}'.$$

Example

By considering the TRS $\mathcal{T}$ of the previous example, we have

$$\mathrm{f}_{\Rightarrow} \cdot \mathsf{c}_2 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \rfloor \mathsf{v}_3 = 2 \, \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 + \mathsf{c}_3 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \mathsf{v}_3 + \mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \rfloor \mathsf{v}_3.$$

Proposition [Compatibilities between rewrite relations and substitutions]

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS, $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms, and σ and σ' be two $\mathcal{S}, \mathsf{V}$-substitutions.

(1) If $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\overline{\sigma} \cdot \mathfrak{t} \Rightarrow \overline{\sigma} \cdot \mathfrak{t}'$.

(2) For any holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\mathfrak{s} \curvearrowleft_{\square} \mathfrak{t} \Rightarrow \mathfrak{s} \curvearrowleft_{\square} \mathfrak{t}'$.

(3) For any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$ and any $v \in \mathsf{V}$, if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\mathfrak{s} \curvearrowleft_{v} \mathfrak{t} \Rightarrow^{*} \mathfrak{s} \curvearrowleft_{v} \mathfrak{t}'$.

(4) If, for all $v \in \mathrm{Vars} \cdot \mathfrak{t}$, $\sigma \cdot v \Rightarrow \sigma' \cdot v$, then $\overline{\sigma} \cdot \mathfrak{t} \Rightarrow^{*} \overline{\sigma'} \cdot \mathfrak{t}$.

Exercise ●○○○○

Define a TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ and provide a nontrivial example for each property (1), (2), (3), and (4) of the previous proposition.

Exercise ●○○○○

Build an example of a ‘‘TRS’’ $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ such that $\mathcal{T}$ does not satisfy (T1) and such that $\mathcal{T}$ does not satisfy Property (1) of the previous proposition.

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7. Termination

7.1. Normal forms and pattern avoidance

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. When $\mathfrak{t}$ is not a factor of $\mathfrak{t}'$, $\mathfrak{t}'$ *avoids* $\mathfrak{t}$.

By extension, for any set X of $\mathcal{S}, \mathsf{V}$-terms, an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}'$ *avoids* X if for any $\mathfrak{t} \in X$, $\mathfrak{t}'$ avoids $\mathfrak{t}$.

Examples

On the signature $\mathcal{S}_{\mathbb{N}^2}$ and the set of variables $\mathsf{V}_{\mathbb{N}}$, let $X := \{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1, \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3, \mathsf{c}_3 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3 \rfloor \mathsf{v}_4\}$.

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

- ☐ $\mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \rfloor \lfloor \mathsf{c}_2 \mathsf{v}_4 \mathsf{v}_5 \rfloor$ avoids X;
- ☐ $\mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \rfloor \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \rfloor$ does not avoid X;
- ☐ $\mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \rfloor \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_2 \rfloor$ avoids X;
- ☐ $\mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_1 \lfloor \mathsf{v}_2 \mathsf{v}_3 \mathsf{v}_1 \rfloor \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_4 \rfloor \rfloor \rfloor$ avoids X.

Exercise ●○○○○○

On the signature $\mathcal{S}_{\mathbb{N}^2}$ and the set of variables $\mathsf{V}_{\mathbb{N}}$, let $X := \{\mathsf{c}_n \mathsf{v}_1 \dots \mathsf{v}_1 : n \in \mathbb{N}\}$. Give an example of a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of constant length 7 avoiding the set X and an example of a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of constant length 5 which does not avoid X.

Theorem [Normal forms and avoidance]

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. The set of normal forms of $\mathcal{T}$ is the set of $\mathcal{S}, \mathrm{V}$-terms avoiding $\mathrm{L} \cdot \mathcal{T}$.

Proof. Let $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term. The property for $\mathfrak{t}$ to be a normal form is equivalent to the fact that there is no $\mathcal{S}, \mathrm{V}$-term $\mathfrak{t}'$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$. By Propositions [Factors of $\mathcal{S}, \mathrm{V}$-terms] and [Rewrite relation of a TRS], this is equivalent to the fact that $\mathfrak{t}$ has no factor $\mathfrak{s}$ such that $\mathfrak{s}$ is the left-hand side of a rewrite rule of $\mathcal{T}$. This property is finally equivalent to the fact that $\mathfrak{t}$ avoids $\mathrm{L} \cdot \mathcal{T}$.

Examples

- ☐ The normal forms of the TRS Assoc avoid $\{m \llcorner m12 \lrcorner 3\}$.
- ☐ The normal forms of the TRS Assoc_2 avoid $\{m \llcorner m12 \lrcorner 3, \; m'1 \llcorner m'23 \lrcorner\}$.
- ☐ The normal forms of the TRS Eq avoid $\{e11\}$.
- ☐ the normal forms of the TRS NatAdd avoid $\{a1z, \; a1 \llcorner s2 \lrcorner\}$.

Let us consider the following combinatorial problem consisting of the following steps:

1. consider a signature $\mathcal{S}$, a set of variables V and a set X of $\mathcal{S}, \mathsf{V}$-terms;
2. define Y as the set of $\mathcal{S}, \mathsf{V}$-terms avoiding X;
3. let Y' be the subset of Y of $\mathcal{S}, \mathsf{V}$-terms satisfying some property (like being ground, linear, or having variables in a finite subset V' of V);
4. consider a rank function rk (like ℓ, ℓ_{cns}, or ℓ_{var}) so that (Y', rk) is combinatorial;
5. describe the integer sequence of $\mathcal{G}$.

Examples

Let us consider the enumeration of graded sets $\mathcal{G}$ of planar labeled $\mathcal{S}_{\mathbb{N}^2}$-terms where rank function is ℓ_{var} and avoiding some sets X of labeled $\mathcal{S}_{\mathbb{N}^2}$-terms:

- ☐ if $X = \left\{\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \; \mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3, \; \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \; \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3\right\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 2, 4, 8, 16, 32, 64, 128 (powers of 2, A000079);
- ☐ if $X = \{\mathsf{c}_{2} \lfloor \mathsf{c}_{2} 12 \rfloor 3, \; \mathsf{c}_{2} \lfloor \mathsf{c}_{3} 123 \rfloor 4, \; \mathsf{c}_{3} \lfloor \mathsf{c}_{2} 12 \rfloor 34, \; \mathsf{c}_{3} \lfloor \mathsf{c}_{3} 123 \rfloor 45\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 1, 2, 4, 9, 21, 51, 127 (Motzkin numbers, A001006);
- ☐ if $X = \left\{\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \; \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \; \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \; \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,1} 2 \lfloor \mathsf{c}_{2,1} 34 \rfloor \rfloor\right\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 2, 5, 13, 35, 96, 267, 750 (directed animals, A005773).

There are some known results for avoidance of sets of

- **binary trees** [E. S. Rowland, Pattern avoidance in binary trees, 2010];
- **ternary trees** [Gabriel, K. Peske, L. Pudwell, S. Tay, Pattern avoidance in ternary trees, 2012];
- **planar labeled $\mathcal{S}$-terms of constant length 2** [S. F. Parker, The combinatorics of functional composition and inversion, 1993], [J.-L. Loday, Inversion of integral series enumerating planar trees, 2005];
- **planar labeled $\mathcal{S}$-terms** [A. Khoroshkin and D. Piontkovski, On generating series of finitely presented operads, 2015], [S. Giraudo, Tree series and pattern avoidance in syntax trees, 2020].

Term series and their grafting operations are powerful tools to describe the characteristic series of such $\mathcal{S}, \mathrm{V}$-terms.

Exercise ○○○○○

Given a signature $\mathcal{S}$, a set of variables V, and a set of $\mathcal{S}, \mathrm{V}$-terms X, provide a systems of equations for the characteristic series of the $\mathcal{S}, \mathrm{V}$-terms avoiding X.

The main difficulty of this research question comes from the avoidance of $\mathcal{S}, \mathrm{V}$-terms which are not linear.