Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m and $\to$ is defined through the rewrite rule $r_1 := m \underbracket{m \underbracket{12} 3} 4 \to m 1 \underbracket{m 2 \underbracket{m 3 4}}$ [C. Chenavier, C. Cordero, S. Giraudo, Quotients of the magmatic operad: lattice structures and convergent rewrite systems, 2019]. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical term associated with the critical data $(r_1, 1, r_1)$ of $\mathcal{T}$ is $m \underbracket{m \underbracket{m \underbracket{m 1^{(1)} 2^{(1)}} 3^{(1)}} 4^{(1)}} 4$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m \underbracket{m 1^{(1)} 2^{(1)}} \underbracket{m 3^{(1)} \underbracket{m 4^{(1)} 4}}$ and $\mathfrak{s}_2 := m \underbracket{m 1^{(1)} \underbracket{m 2^{(1)} \underbracket{m 3^{(1)} 4^{(1)}}}} 4$. These two $\mathcal{S}, \mathrm{C}\cdot\mathbb{N}$-terms are normal forms of $\mathrm{C}\cdot\mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_2 := (\mathfrak{s}_1, \mathfrak{s}_2)$.

Let the TRS $\mathcal{T}' := (\mathcal{S}, \mathrm{C}\cdot\mathbb{N}, \{r_1, r_2\})$. The critical term associated with the critical data $(r_1, 11, r_1)$ of $\mathcal{T}'$ is $m \underbracket{m \underbracket{m \underbracket{m \underbracket{m 1^{(1)} 2^{(1)}} 3^{(1)}} 4^{(1)}} 3} 4$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m \underbracket{m \underbracket{m 1^{(1)} 2^{(1)}} 3^{(1)}} \underbracket{m 4^{(1)} \underbracket{m 23}}$ and $\mathfrak{s}_2 := m \underbracket{m \underbracket{m 1^{(1)} \underbracket{m 2^{(1)} \underbracket{m 3^{(1)} 4^{(1)}}}} 3} 4$. We have that $\mathfrak{s}'_1 := m 1^{(1)} \underbracket{m 2^{(1)} \underbracket{m 3^{(1)} \underbracket{m 4^{(1)} \underbracket{m 34}}}}$ and, by using the rewrite rule r_2, $\mathfrak{s}'_2 := m 1^{(1)} \underbracket{m 2^{(1)} \underbracket{m \underbracket{m 3^{(1)} \underbracket{m 4^{(1)} 3}} 4}}$ are respective normal forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathcal{T}'$. Therefore, $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ adds to $\{r_1, r_2\}$ the new rewrite rule $r_3 := (\mathfrak{s}'_1, \mathfrak{s}'_2)$.

Exercise ●●●○○

Let us consider the previous TRS $\mathcal{T}$. Show that $\mathcal{T}$ leads to a convergent TRS formed by eleven rewrite rules.

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9. Universal algebra and clones

/ Universal algebra and clones

9.1. Varieties of algebras

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

An $\mathcal{S}$-algebra $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ is an $\mathcal{S}$-*subalgebra* of $\mathcal{A}$ if $X' \subseteq X$ and for any $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, and any $x'_1, \dots, x'_n \in X'$,

$$\mathrm{op}'{\cdot}c{\cdot}x'_1{\cdot} \cdots {\cdot}x'_n = \mathrm{op}{\cdot}c{\cdot}x'_1{\cdot} \cdots {\cdot}x'_n.$$

Examples

Let MagC be the signature containing one nullary constant c and one binary constant m.

- ☐ The triple $\mathcal{A} := (\mathcal{P}{\cdot}\mathbb{Z}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op}{\cdot}c := \mathbb{Z}$ and $\mathrm{op}{\cdot}m{\cdot}S_1{\cdot}S_2 := S_1 \cap S_2$ is a MagC-algebra.

  The triple $(\mathcal{P}{\cdot}\mathbb{N}, \mathrm{MagC}, \mathrm{op})$ where op is the same function op as before, but restricted on $\mathcal{P}{\cdot}\mathbb{N}$, is a MagC-subalgebra of $\mathcal{A}$.

- ☐ The triple $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op}{\cdot}c := 0$ and $\mathrm{op}{\cdot}m{\cdot}n_1{\cdot}n_2 := n_1 + n_2$ is a MagC-algebra.

  The triple $(\{2n : n \in \mathbb{N}\}, \mathrm{MagC}, \mathrm{op})$ where op is the same function op as before, but restricted on even natural numbers, is a MagC-subalgebra of $\mathcal{A}$.

- ☐ By considering the previous MagC-algebra $\mathcal{A}$, the triple $([\![k]\!], \mathrm{MagC}, \mathrm{op})$, $k \in \mathbb{N}$, where op is the same function op as before, but restricted on natural numbers smaller than or equal to k, is **not** a MagC-subalgebra of $\mathcal{A}$ because $\mathrm{op}{\cdot}m$ is not well-defined.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

Given $Y \subseteq X$, the *$\mathcal{S}$-subalgebra of $\mathcal{A}$ generated by Y* is the $\mathcal{S}$-subalgebra $\mathcal{A}^{(Y)}$ of $\mathcal{A}$ whose underlying set contains Y and is minimal w.r.t. inclusion among the underlying sets of all $\mathcal{S}$-subalgebras of $\mathcal{A}$ which satisfy this property.

When $\mathcal{A} = \mathcal{A}^{(Y)}$, Y is a *generating set* of $\mathcal{A}$.

Examples

- ☐ Let the MagC-algebra $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 0$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$.

  We have $\mathcal{A}^{(\{1\})} = \mathcal{A}$. This follows by induction on $\mathbb{N}$. First, we have $0 \in \mathcal{A}^{(\{1\})}$ because $0 = \mathrm{op} \cdot c$. We have also $1 \in \mathcal{A}^{(\{1\})}$ trivially because 1 is an element of the generating set. Moreover, for any $n \in \mathbb{N} \setminus \{0\}$, $n = \mathrm{op} \cdot m \cdot \llcorner n - 1 \lrcorner \cdot 1$, so that, by induction hypothesis, $n \in \mathcal{A}^{(\{1\})}$.

- ☐ Let the MagC-algebra $\mathcal{A} := (\mathbb{Z}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 1$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 \times n_2$.

  The MagC-subalgebra $\mathcal{A}^{(\{2,3\})}$ of $\mathcal{A}$ has as underlying set $\left\{2^{k_1} 3^{k_2} : k_1, k_2 \in \mathbb{N}\right\}$.

Let I be a set and $(\mathcal{A}_i)_{i \in I}$ be a family such that for any $i \in I$, $\mathcal{A}_i := (X_i, \mathcal{S}, \mathrm{op}_i)$ is an $\mathcal{S}$-algebra.

The *product* of $(\mathcal{A}_i)_{i \in I}$ is the $\mathcal{S}$-algebra $\prod_{i \in I} \mathcal{A}_i := (X, \mathcal{S}, \mathrm{op})$ such that $X := \prod_{i \in I} X_i$ and for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any $\left(x_i^{(1)}\right)_{i \in I}, \dots, \left(x_i^{(n)}\right)_{i \in I} \in X$,

$$\mathrm{op} \cdot c \cdot \left(x_i^{(1)}\right)_{i \in I} \cdot \cdots \cdot \left(x_i^{(n)}\right)_{i \in I} := \left(\mathrm{op}_i \cdot c \cdot x_i^{(1)} \cdot \cdots \cdot x_i^{(n)}\right)_{i \in I}.$$

Example

Let the MagC-algebras $\mathcal{A}_1 := (\mathbb{N}, \mathrm{MagC}, \mathrm{op}_1)$, where $\mathrm{op}_1 \cdot c := 0$ and $\mathrm{op}_1 \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$, and $\mathcal{A}_2 := (\mathbb{Z}, \mathrm{MagC}, \mathrm{op}_2)$, where $\mathrm{op}_2 \cdot c := 1$ and $\mathrm{op}_2 \cdot m \cdot n_1 \cdot n_2 := n_1 \times n_2$.

By setting $\mathcal{A} := \prod_{i \in [2]} \mathcal{A}_i = (X, \mathcal{S}, \mathrm{op})$, we have

- ☐ $X = \mathbb{N} \times \mathbb{Z}$;
- ☐ $\mathrm{op} \cdot c = (0, 1)$;
- ☐ For any $\left(n_1^{(1)}, n_2^{(1)}\right), \left(n_1^{(2)}, n_2^{(2)}\right) \in X$, $\mathrm{op} \cdot m \cdot \left(n_1^{(1)}, n_2^{(1)}\right) \cdot \left(n_1^{(2)}, n_2^{(2)}\right) = \left(n_1^{(1)} + n_1^{(2)}, n_2^{(1)} \times n_2^{(2)}\right)$.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ and $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ be two $\mathcal{S}$-algebras.

An *$\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$* is a function $\phi : X \to X'$ such that for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any $x_1, \dots, x_n \in X$,

$$\phi \cdot \underbracket{\mathrm{op} \cdot c \cdot x_1 \cdot \ \cdots \ \cdot x_n} = \mathrm{op}' \cdot c \cdot \underbracket{\phi \cdot x_1} \cdot \ \cdots \ \cdot \underbracket{\phi \cdot x_n}.$$

Example

Let the MagC-algebras $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \epsilon$ and $\mathrm{op} \cdot m \cdot w_1 \cdot w_2 := w_1 \centerdot w_2$, and $\mathcal{A}' := (\mathbb{N}, \mathrm{MagC}, \mathrm{op}')$, where $\mathrm{op}' \cdot c := 0$ and $\mathrm{op}' \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$.

The function ℓ, which sends any $w \in \{a, b\}^*$ to the length of w, is an MagC-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$. Indeed,

$$\ell \cdot \underbracket{\mathrm{op} \cdot c} = \ell \cdot \epsilon = 0 = \mathrm{op}' \cdot c$$

and, for any $w_1, w_2 \in \{a, b\}^*$,

$$\ell \cdot \underbracket{\mathrm{op} \cdot m \cdot w_1 \cdot w_2} = \ell \cdot \underbracket{w_1 \centerdot w_2} = \ell \cdot w_1 + \ell \cdot w_2 = \mathrm{op}' \cdot m \cdot \underbracket{\ell \cdot w_1} \cdot \underbracket{\ell \cdot w_2}.$$

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

An equivalence relation $\equiv$ on X is a *congruence on* $\mathcal{A}$ if for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, any $x_1, \dots, x_i, x_i', \dots x_n \in X$, $i \in [n]$, $x_i \equiv x_i'$ implies

$$\mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_i \cdot \cdots \cdot x_n \equiv \mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_i' \cdot \cdots \cdot x_n.$$

The $\equiv$-equivalence class of $x \in X$ in $\mathcal{A}$ is denoted by $[x]_{\equiv}$.

The *quotient* of $\mathcal{A}$ by a congruence $\equiv$ on $\mathcal{A}$ is the $\mathcal{S}$-algebra $\mathcal{A}/\equiv := (X/\equiv, \mathcal{S}, \mathrm{op}/\equiv)$ such that $\mathrm{op}/\equiv$ is defined, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, by

$$\mathrm{op}/\equiv \cdot c \cdot [x_1]_{\equiv} \cdot \cdots \cdot [x_n]_{\equiv} := [\mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_n]_{\equiv}.$$

Example

Let the MagC-algebra $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \epsilon$ and $\mathrm{op} \cdot m \cdot w_1 \cdot w_2 := w_1 \mathbin{\mathbf{.}} w_2$.
Let $\equiv$ be the equivalence relation on $\{a, b\}^*$ such that, for any $w_1, w_2 \in \{a, b\}^*$, $w_1 \equiv w_2$ if $\ell_a \cdot w_1 = \ell_a \cdot w_2$ and $\ell_b \cdot w_1 = \ell_b \cdot w_2$. It is easy to check that $\equiv$ is a congruence of $\mathcal{A}$.
The underlying set $\{a, b\}^*/\equiv$ of $\mathcal{A}/\equiv$ admits $\{a^{n_a} b^{n_b} : n_a, n_b \in \mathbb{N}\}$ as set of representatives.
Moreover, $\mathrm{op}/\equiv \cdot c = [\epsilon]_{\equiv}$ and, for any $n_a, n_b, n_a', n_b' \in \mathbb{N}$,

$$\mathrm{op}/\equiv \cdot m \cdot [a^{n_a} b^{n_b}]_{\equiv} \cdot \left[a^{n_a'} b^{n_b'}\right]_{\equiv} = \left[a^{n_a + n_a'} b^{n_b + n_b'}\right]_{\equiv}.$$

Let X and X' be two sets and $\phi : X \to X'$ be a function.

- ☐ The *kernel* of ϕ is the equivalence relation $\mathrm{Ker}\cdot\phi$ on X defined by $(x_1, x_2) \in \mathrm{Ker}\cdot\phi$ if $\phi\cdot x_1 = \phi\cdot x_2$.
- ☐ The *image* of ϕ is the set $\mathrm{Im}\cdot\phi := \{x' \in X' : \text{ there exists } x \in X \text{ such that } \phi\cdot x = x'\}$.

Theorem [First Isomorphism Theorem]

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ be two $\mathcal{S}$-algebras, and ϕ be an $\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$.

1. The equivalence relation $\mathrm{Ker}\cdot\phi$ is a congruence on $\mathcal{A}$.
2. The triple $\mathcal{A}'' := (\mathrm{Im}\cdot\phi, \mathcal{S}, \mathrm{op}'')$ is an $\mathcal{S}$-subalgebra of $\mathcal{A}'$, where, for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, $\mathrm{op}''\cdot c$ is the restriction of $\mathrm{op}'\cdot c$ as an n-operation on $\mathrm{Im}\cdot\phi$.
3. The function $\bar{\phi} : X/\mathrm{Ker}\cdot\phi \to \mathrm{Im}\cdot\phi$ defined for any $x \in X$ by $\bar{\phi}\cdot[x]_{\mathrm{Ker}\cdot\phi} := \phi\cdot x$ is an $\mathcal{S}$-algebra isomorphism between $\mathcal{A}/\mathrm{Ker}\cdot\phi$ and $\mathcal{A}''$.

Exercise ●●●○○

Prove Theorem [First Isomorphism Theorem].

Let $\mathcal{S}$ be a signature and $\mathfrak{C}$ be a class of $\mathcal{S}$-algebras.

The class $\mathfrak{C}$ is

- ☐ *closed under homomorphic images* when for any $\mathcal{A} \in \mathfrak{C}$ and any $\mathcal{S}$-algebra $\mathcal{A}'$, if there is a surjective $\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$, then $\mathcal{A}' \in \mathfrak{C}$;
- ☐ *closed under subalgebras* when for any $\mathcal{A} \in \mathfrak{C}$, any $\mathcal{S}$-subalgebra of $\mathcal{A}$ belongs to $\mathfrak{C}$;
- ☐ *closed under products* when for any set I and any family $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{S}$-algebras belonging to $\mathfrak{C}$, $\prod_{i \in I} \mathcal{A}_i \in \mathfrak{C}$.

Note that by Theorem [First Isomorphism Theorem], if $\mathfrak{C}$ is closed under homomorphic images, then for any $\mathcal{A} \in \mathfrak{C}$ and any congruence $\equiv$ on $\mathcal{A}$, the quotient $\mathcal{A}/\equiv$ belongs to $\mathfrak{C}$.

Note also that the closure under homomorphic images implies that if $\mathcal{A} \in \mathfrak{C}$, then any $\mathcal{S}$-algebra isomorphic to $\mathcal{A}$ belongs to $\mathfrak{C}$.

Definition

Let $\mathcal{S}$ be a signature. A *variety* of $\mathcal{S}$-algebras is a nonempty class $\mathfrak{C}$ of $\mathcal{S}$-algebras which is closed under homomorphic images, subalgebras, and products.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The $\mathcal{S}, \mathsf{V}$*-term algebra* is the $\mathcal{S}$-algebra $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} := (\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \mathcal{S}, \mathrm{op})$ such that op is defined, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$, by

$$\mathrm{op} \cdot c \cdot \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_n := c \, \mathfrak{t}_1 \dots \mathfrak{t}_n .$$

The $\mathcal{S}, \emptyset$-term algebra is the *ground* $\mathcal{S}$*-term algebra*. Note that if $\mathcal{S} \cdot 0 = \emptyset$, then the underlying set of $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ is empty.

Example

The $\mathrm{MagC}, \mathbb{N}$-term algebra $\mathbf{T} \cdot \mathrm{MagC} \cdot \mathbb{N}$ admits the set of $\mathrm{MagC}, \mathbb{N}$-terms. Moreover, we have for instance

$$\mathrm{op} \cdot c = c$$

and

$$\mathrm{op} \cdot m \cdot m1\underbracket{mc2} \cdot m23 = m\underbracket{m1\underbracket{mc2}}\underbracket{m23}.$$

Besides, the underlying set of the ground $\mathcal{S}$-term algebra is

$$X := \{c, mcc, m\underbracket{mcc}c, mc\underbracket{mcc}, \dots\}$$

and the graded set (X, ℓ_c) is a combinatorial graded set and its integer sequence is the one of Catalan numbers.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\iota : \mathsf{V} \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ be the function such that for any $v \in \mathsf{V}$, $\iota \cdot v$ is the leaf decorated by v.

Theorem [Free $\mathcal{S}$-algebras]

For any signature $\mathcal{S}$, any set of variables V, any $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, and any V, X-assignment α, there exists a unique $\mathcal{S}$-algebra morphism ϕ from $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V}$ to $\mathcal{A}$ such that $\alpha = \phi \circ \iota$.

The class of $\mathcal{S}$-algebras together with $\mathcal{S}$-algebra morphisms forms a category.

Theorem [Free $\mathcal{S}$-algebras] says that $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V}$ is a free object in this category.

For any set X, the *empty X-assignment* is the $\emptyset, X$-assignment $\emptyset$ having an empty domain.

For any $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, $\mathrm{ev}_{\mathcal{A}, \emptyset}$ is the unique $\mathcal{S}$-algebra morphism from $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ to $\mathcal{A}$. Therefore, $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ is an initial object in the category of $\mathcal{S}$-algebras.

/ Universal algebra and clones

9.2. Equational presentations

Definition

An $\mathcal{S}, \mathsf{V}$-*equational presentation* is a triple $(\mathcal{S}, \mathsf{V}, \sim)$ where

- ☐ $\mathcal{S}$ is a signature, called the *underlying signature*;
- ☐ V is a set of variables, called the *underlying set of variables*;
- ☐ $\sim$ is a binary relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$, called the *elementary identity relation*.

Remark that this definition looks like the one of TRSs but there is no condition on $\sim$ here.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation. If t and t' are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathsf{t} \sim \mathsf{t}'$, then the pair $(\mathsf{t}, \mathsf{t}')$ is an *elementary identity* of $\mathcal{E}$.

Examples

- ☐ Let $\mathbf{Monoids} := (\mathrm{MagC}, \mathbb{N}, \sim)$ be the $\mathrm{MagC}, \mathbb{N}$-equational presentation such that $\sim$ is defined by $m\underbracket{m12}3 \sim m1\underbracket{m23}$, $m1c \sim 1$, and $mc1 \sim 1$.

- ☐ Let $\mathbf{BSLattices} := (\mathrm{MagC}, \mathbb{N}, \sim)$ be the $\mathrm{MagC}, \mathbb{N}$-equational presentation such that $\sim$ is defined by $m\underbracket{m12}3 \sim m1\underbracket{m23}$, $m1c \sim 1$, $mc1 \sim 1$, $m12 \sim m21$, and $m11 \sim 1$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra. Two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *$\mathcal{A}$-equivalent* if for any V, X-assignment α, $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'$.

An *algebra over $\mathcal{E}$* is an $\mathcal{S}$-algebra $\mathcal{A}$ such that for any elementary identity $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{E}$, $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{A}$-equivalent.

Examples

- ☐ Let the MagC-algebra $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 0$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$.

  The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m1\underbracket{m12}$ and $\mathfrak{t}' := m\underbracket{m21}1$ are $\mathcal{A}$-equivalent. Indeed, for any $\mathbb{N}, \mathbb{N}$ assignment α, by setting $n_1 := \alpha \cdot 1$ and $n_2 := \alpha \cdot 2$,

  $$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} = n_1 + (n_1 + n_2) = 2n_1 + n_2 = (n_2 + n_1) + n_1 = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'.$$

- ☐ The class of algebras over **Monoids** is the class of monoids.
- ☐ The class of algebras over **BSLattices** is the class of bounded semilattices.

Here is a list of some important varieties appearing frequently in algebraic combinatorics.

- The variety of monoids, described by the equational presentation **Monoids**.
- The variety of bounded semilattices, described by the equational presentation **BSLattices**.
- The variety of groups, described by the equational presentation $\mathbf{Groups} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \underbracket{m12} 3 \sim m1 \underbracket{m23}$, $me1 \sim 1$, $m1e \sim 1$, $m \underbracket{i1} 1 \sim e$, and $m1 \underbracket{i1} \sim e$.
- The variety of idempotent semigroups (also called *bands*), described by the equational presentation $\mathbf{Bands} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains one binary constant m and $\sim$ is defined by $m \underbracket{m12} 3 \sim m1 \underbracket{m23}$ and $m11 \sim 1$.
- The variety of *duplicial algebras* [C. Brouder, A. Frabetti, QED Hopf algebras on planar binary trees, 2003], described by the equational presentation $\mathbf{DuplicialAlgebras} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains two binary constant $\ll$ and $\gg$, and $\sim$ is defined by $\ll \underbracket{\ll 12} 3 \sim \ll 1 \underbracket{\ll 23}$, $\gg \underbracket{\gg 12} 3 \sim \gg 1 \underbracket{\gg 23}$, and $\gg \underbracket{\ll 12} 3 \sim \ll 1 \underbracket{\gg 23}$.
- The variety of *nonassociative permutative algebras* [M. Livernet, A rigidity theorem for pre-Lie algebras, 2006], described by the equational presentation $\mathbf{NAPAlgebras} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains one binary constant g, and $\sim$ is defined by $\mathsf{g} \underbracket{\mathsf{g}12} 3 \sim \mathsf{g} \underbracket{\mathsf{g}13} 2$.

Theorem [Birkhoff's Variety Theorem]

Let $\mathcal{S}$ be a signature and V be an infinite set of variables. A nonempty class $\mathfrak{C}$ of $\mathcal{S}$-algebras is a variety iff there exists an $\mathcal{S}, \mathsf{V}$-equational presentation $\mathcal{E}$ such that $\mathfrak{C}$ is the class of algebras over $\mathcal{E}$.

This result comes from [G. Birkhoff, On the Structure of Abstract Algebras, 1935] and is also known as the Birkhoff HSP Theorem.

When $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is an $\mathcal{S}, \mathsf{V}$-equational presentation, by [Birkhoff's Variety Theorem], the class of algebras over $\mathcal{E}$ is a variety $\mathcal{V}$. We call $\mathcal{V}$ the *variety of* $\mathcal{E}$.

Examples

- ☐ Since the class $\mathfrak{C}$ of monoids is closed under homomorphic images, subalgebras, and products, $\mathfrak{C}$ is a variety. By Theorem [Birkhoff's Variety Theorem], there exists an $\mathcal{S}, \mathsf{V}$-equational presentation of $\mathfrak{C}$. This is the $\mathrm{MagC}, \mathbb{N}$-equational presentation **Monoids**.
- ☐ By Theorem [Birkhoff's Variety Theorem], **BSLattices** is an $\mathrm{MagC}, \mathbb{N}$-equational presentation of a variety. This is the variety of bounded semilattices.