Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

Two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{E}$*-semantically equivalent* if for any algebra $\mathcal{A}$ over $\mathcal{E}$, $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{A}$-equivalent. This property is denoted by $\mathfrak{t} \approx_{\mathcal{E}} \mathfrak{t}'$.

Examples

- ☐ The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m1\underbracket{m12}$ and $\mathfrak{t}' := m\underbracket{m21}1$ are not $\mathbf{Monoids}$-semantically equivalent.

  Indeed, consider the MagC-algebra $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \epsilon$ and $\mathrm{op} \cdot m \cdot w_1 \cdot w_2 := w_1 \centerdot w_2$ is such that, by setting α as an $\mathbb{N}, \{a, b\}^*$-assignment satisfying $\alpha \cdot 1 = a$ and $\alpha \cdot 2 = b$, $\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t} = aab \neq baa = \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}'$.
- ☐ The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m\underbracket{m12}\underbracket{m21}$ and $\mathfrak{t}' := m12$ are $\mathbf{BSLattices}$-semantically equivalent.
- ☐ The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := mc1$ and $\mathfrak{t}' := c$ are not $\mathbf{BSLattices}$-semantically equivalent.

Exercise ●○○○○

Prove the last two properties of the previous examples.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

When $\sim$ is an elementary rewrite relation, $\mathcal{E}$ is *TRS-like*. In this case, $\mathcal{E}$ is a TRS.

Examples

- ☐ The equational presentations **Monoids**, **BSLattices**, and **Groups** are TRS-like.
- ☐ The equational presentation $(\mathrm{MagC}, \mathbb{N}, \sim)$ such that $\sim$ is defined by $1 \sim m11$ is not TRS-like.
- ☐ The equational presentation $(\mathrm{MagC}, \mathbb{N}, \sim)$ such that $\sim$ is defined by $mcc \sim m1c$ is not TRS-like.

When $\mathcal{E}$ is TRS-like, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *$\mathcal{E}$-syntactically equivalent* if $\mathfrak{t} \equiv \mathfrak{t}'$ where $\equiv$ is the **convertibility relation** of $\mathcal{E}$.

Example

Let the $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m\underline{mc1}\underline{m3\underline{m\underline{m3c}2}}$ and $\mathfrak{t}' := m\underline{m12}3$. In **BSLattices**, we have

$$\mathfrak{t}' \Rightarrow m1\underline{m23} \Leftarrow m\underline{mc1}\underline{m23} \Rightarrow m\underline{mc1}\underline{m32} \Leftarrow m\underline{mc1}\underline{m\underline{m33}2} \Leftarrow m\underline{mc1}\underline{m\underline{m3\underline{m3c}}2} \Rightarrow \mathfrak{t}$$

so that $\mathfrak{t}$ and $\mathfrak{t}'$ are **BSLattices**-syntactically equivalent.

Theorem [Equivalence of the $\mathcal{E}$-semantic and $\mathcal{E}$-syntactic relations]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a TRS-like $\mathcal{S}, \mathsf{V}$-equational presentation. The $\mathcal{E}$-semantic equivalence relation and the $\mathcal{E}$-syntactic equivalence relation coincide.

This result is known as the **Birkhoff Theorem on identities** [G. Birkhoff, On the Structure of Abstract Algebras, 1935].

Due to this property, given a TRS-like equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ *$\mathcal{E}$-equivalent* if $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{E}$-semantically equivalent, or, equivalently, $\mathcal{E}$-syntactically equivalent. This property is denoted by $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$.

Exercise ●●○○○○

Let the $\mathcal{S}, \mathbb{N}$-equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N}, \sim)$ such that $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant f, and $\sim$ is defined by $f1\underbracket{f23} \sim f\underbracket{f12}3$, $fe1 \sim 1$, and $f1\underbracket{i1} \sim e$.

Show that the $\mathcal{S}, \mathbb{N}$-terms $f1e$ and 1 are not $\mathcal{E}$-equivalent.

Proposition [$\mathcal{S}$-equivalence and quotients of free $\mathcal{S}, \mathsf{V}$-term algebras]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a equational presentation. The $\mathcal{E}$-equivalence relation $\approx_{\mathcal{E}}$ is a congruence on $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an equational presentation. Let $\bar{\iota} : \mathsf{V} \to \lfloor \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \rfloor / \approx_{\mathcal{E}}$ be the function such that for any $v \in \mathsf{V}$, $\bar{\iota} \cdot v := [\iota \cdot v]_{\approx_{\mathcal{E}}}$.

Theorem [Free algebras over equational presentations]

For any equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$, any algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ over $\mathcal{E}$, and any V, X-assignment α, there exists a unique $\mathcal{S}$-algebra morphism ϕ from $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} / \approx_{\mathcal{E}}$ to $\mathcal{A}$ such that $\alpha = \phi \circ \bar{\iota}$.

The class of algebras over $\mathcal{E}$ together with $\mathcal{S}$-algebra morphisms forms a category.

Theorem [Free algebras over equational presentations] says that $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} / \approx_{\mathcal{E}}$ is a free object in this category.

For any algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ over $\mathcal{E}$, the function $[\mathfrak{t}]_{\approx_{\mathcal{E}}} \mapsto \mathrm{ev}_{\mathcal{A}, \emptyset} \cdot \mathfrak{t}$ is the unique $\mathcal{S}$-algebra morphism from $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset / \approx_{\mathcal{E}}$ to $\mathcal{A}$. Therefore, $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset / \approx_{\mathcal{E}}$ is an initial object in the category of algebras over $\mathcal{E}$.

9.3. Word problem

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *word problem* on an equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is the **decision problem** whose input is two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and whose question is whether $\mathfrak{t} \approx_{\mathcal{E}} \mathfrak{t}'$.

Theorem [Undecidability of the word problem]

There exist equational presentations $\mathcal{E}$ such that the word problem on $\mathcal{E}$ is undecidable.

Proof. Let the equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ is the signature containing two nullary constants K and S, and a binary constant a, and $\sim$ is defined by $\mathsf{a}\underbracket{\mathsf{aK}1}2 \sim 1$ and $\mathsf{a}\underbracket{\mathsf{a}\underbracket{\mathsf{aS}1}2}3 \sim \mathsf{a}\underbracket{\mathsf{a}13}\underbracket{\mathsf{a}23}$. This equational presentation, coming from **combinatory logic**, is known to have an undecidable word problem.

Example

An instance of the word problem on **Groups** is formed by the $\mathcal{S}, \mathbb{N}$-terms $\mathfrak{t} := i\underbracket{m12}$ and $\mathfrak{t}' := m\underbracket{i2}\underbracket{i1}$. We can check that the answer is yes since $\mathfrak{t} \approx_{\mathbf{Groups}} \mathfrak{t}'$.

Theorem [Convergence and decidability of words problems]

Let $\mathcal{E}$ be a TRS-like equational presentation. If $\mathcal{E}$ is convergent and its elementary rewrite relation is finite, then the word problem on $\mathcal{E}$ is decidable.

When $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is a convergent TRS-like equational presentation, an algorithm to decide for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ whether $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$ consists in computing the unique normal form of $\mathfrak{t}$ and of $\mathfrak{t}'$ of $\mathcal{E}$ and checking if they are equal.

Example

By considering the convergent TRS-like equational presentation $\mathbf{Monoids}$, let $\mathfrak{t} := m\underbracket{m1\underbracket{mc2}}\underbracket{m13}$ and $\mathfrak{t}' := m\underbracket{m12}\underbracket{m\underbracket{m1c}\underbracket{mc3}}$. By denoting by $\Rightarrow$ the rewrite relation of $\mathbf{Monoids}$, we have

$$\mathfrak{t} \Rightarrow m\underbracket{m12}\underbracket{m13} \Rightarrow m1\underbracket{m2\underbracket{m13}} := \mathfrak{s}$$

and

$$\mathfrak{t}' \Rightarrow m\underbracket{m12}\underbracket{m\underbracket{m1c}\underbracket{mc3}} \Rightarrow m\underbracket{m12}\underbracket{m1\underbracket{mc3}} \Rightarrow m\underbracket{m12}\underbracket{m13} \Rightarrow m1\underbracket{m2\underbracket{m13}} = \mathfrak{s}.$$

Since $\mathfrak{s}$ is a normal form of $\mathbf{Monoids}$ and $\mathfrak{s} \in \mathfrak{t} \Downarrow \mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathbf{Monoids}} \mathfrak{t}'$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a convergent TRS-like equational presentation.

The *algebra of normal forms of* $\mathcal{E}$ is the $\mathcal{S}$-algebra $\mathcal{A}_{\Rrightarrow} \cdot \mathcal{E} := (X, \mathcal{S}, \mathrm{op})$ such that X is the set of normal forms of $\mathcal{E}$, and for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any normal forms $\mathfrak{t}_1$, ..., $\mathfrak{t}_n$ of $\mathcal{E}$, $\mathrm{op} \cdot c \cdot \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_n$ is the unique normal form of the $\mathcal{S}, \mathsf{V}$-term $c\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ of $\mathcal{E}$.

Theorem [Algebras of normal forms and free algebras]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a convergent TRS-like equational presentation and X be the underlying set of $\mathcal{A}_{\Rrightarrow} \cdot \mathcal{E}$. Let $\phi : \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} / \equiv_{\mathcal{E}} \to X$ be the function such that $\phi \cdot [\mathfrak{t}]_{\equiv_{\mathcal{E}}}$ is the unique normal form in the future of $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ in $\mathcal{E}$. Then, ϕ is an $\mathcal{S}$-algebra isomorphism from $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} / \equiv_{\mathcal{E}}$ to $\mathcal{A}_{\Rrightarrow} \cdot \mathcal{E}$.

Since $\mathcal{E}$ is a convergent TRS, by Proposition [Church-Rosser property], the function ϕ of the statement of Theorem [Algebras of normal forms and free algebras] is well-defined.

Example

The MagC-algebra $\mathbf{T} \cdot \mathrm{MagC} \cdot \mathbb{N} / \equiv_{\mathbf{Monoids}}$ is isomorphic to the algebra $\mathcal{A}_{\Rrightarrow} \cdot \mathbf{Monoids}$ of normal forms of $\mathbf{Monoids}$. The underlying set of $\mathcal{A}_{\Rrightarrow} \cdot \mathbf{Monoids}$ is the set of $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is a variable, or $\mathfrak{t} = c$, or $\mathfrak{t} = m i_1 \underbracket{m i_2 \underbracket{m i_3 \underbracket{\dots \underbracket{m i_n i_{n+1}} \dots}}}$ where $n \geqslant 1$ and for any $j \in [n+1]$, $i_j \in \mathbb{N}$. Moreover, in $\mathcal{A}_{\Rrightarrow} \cdot \mathbf{Monoids}$,

$$\mathrm{op} \cdot m \cdot \underbracket{m 1 \underbracket{m 2 3}} \cdot \underbracket{m 1 \underbracket{m 2 \underbracket{m 3 4}}} = m 1 \underbracket{m 2 \underbracket{m 3 \underbracket{m 1 \underbracket{m 2 \underbracket{m 3 4}}}}}.$$

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10. Clones and varieties

/ Clones and varieties

10.1. Clones

Two varieties may be **equivalent** even if their underlying signatures are different.

Example

Let the equational presentation $\mathbf{PHeaps} := (\mathcal{S}', \mathbb{N}, \sim')$ where $\mathcal{S}'$ is the signature containing a nullary constant e' and a ternary constant p', and $\sim'$ is defined by $p'112 \sim' 2$, $p'122 \sim' 1$, and $p' \llcorner p'123 \lrcorner 45 \sim' p'12 \llcorner p'345 \lrcorner$.

The varieties of **Groups** and the variety of *pointed heaps* of **PHeaps** are equivalent.

Indeed, add to **PHeaps** a unary constant i' and a binary constant m', and set $m'12 \sim' p'1e'2$ and $i'1 \sim' p'e'1e'$. We have for instance

$$m' \llcorner i'1 \lrcorner 1 \approx_{\mathbf{PHeaps}} p' \llcorner i'1 \lrcorner e'1 \approx_{\mathbf{PHeaps}} p' \llcorner p'e'1e' \lrcorner e'1 \approx_{\mathbf{PHeaps}} p'e'1 \llcorner p'e'e'1 \lrcorner \approx_{\mathbf{PHeaps}} p'e'11 \approx_{\mathbf{PHeaps}} e'.$$

Similarly, the five elementary identities of **Groups** hold in **PHeaps** on e', i', and m'.

Conversely, add to **Groups** a nullary constant e' and a ternary constant p', and set $e' \sim e$ and $p'123 \sim m \llcorner m1 \llcorner i2 \lrcorner \lrcorner 3$. We have for instance

$$p'112 \approx_{\mathbf{Groups}} m \llcorner m1 \llcorner i1 \lrcorner \lrcorner 2 \approx_{\mathbf{Groups}} me2 \approx_{\mathbf{Groups}} 2.$$

Similarly, the three elementary identities of **PHeaps** hold in **Groups** on e' and p'.

The variety of *Heaps* (pointed heaps are heaps with distinguished unit element) goes back at [H. Prüfer, Theorie der Abelschen Gruppen. I. Grundeigenschaften, 1924].

In order to explain such equivalences, we need algebraic structures yielding an **invariant** of an equational presentation, independent of the signature. **Abstract clones** are such invariants.

An *abstract clone* (or *clone* for short) is a triple $\left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ where

- ☐ $\mathcal{G}$ is a graded set, called the *underlying set*;
- ☐ for any $n, m \in \mathbb{N}$, $\gamma_{n,m}$ is a function of type

  $$\mathcal{G}{\cdot}n \to \underbrace{\mathcal{G}{\cdot}m \to \cdots \to \mathcal{G}{\cdot}m}_{n \text{ times}} \to \mathcal{G}{\cdot}m,$$

  called the *n, m-superposition function*. In other words, $\gamma_{n,m}{\cdot}x$ is an n-operation on $\mathcal{G}{\cdot}m$ for any $x \in \mathcal{G}{\cdot}n$, $n \in \mathbb{N}$;
- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is an element of $\mathcal{G}{\cdot}n$, called the *i, n-projection*;
- ☐ the following identities hold:
  1. for any $n, m \in \mathbb{N}$, $i \in [n]$, and $x_1, \dots, x_n \in \mathcal{G}{\cdot}m$,

     $$\gamma_{n,m}{\cdot}\mathbb{1}_{i,n}{\cdot}x_1{\cdot} \cdots {\cdot}x_n = x_i;$$

  2. for any $n \in \mathbb{N}$ and $x \in \mathcal{G}{\cdot}n$,

     $$\gamma_{n,n}{\cdot}x{\cdot}\mathbb{1}_{1,n}{\cdot} \cdots {\cdot}\mathbb{1}_{n,n} = x;$$

  3. for any $n, m, k \in \mathbb{N}$, $x \in \mathcal{G}{\cdot}n$, $y_1, \dots, y_n \in \mathcal{G}{\cdot}m$, and $z_1, \dots, z_m \in \mathcal{G}{\cdot}k$,

     $$\gamma_{m,k}{\cdot}\llcorner\gamma_{n,m}{\cdot}x{\cdot}y_1{\cdot} \cdots {\cdot}y_n\lrcorner{\cdot}z_1{\cdot} \cdots z_m = \gamma_{n,k}{\cdot}x{\cdot}\llcorner\gamma_{m,k}{\cdot}y_1{\cdot}z_1{\cdot} \cdots {\cdot}z_m\lrcorner{\cdot} \cdots {\cdot}\llcorner\gamma_{m,k}{\cdot}y_n{\cdot}z_1{\cdot} \cdots {\cdot}z_m\lrcorner{\cdot}$$

Example

Let the clone $\mathrm{SL} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- ☐ for any $n \in \mathbb{N}$, $\mathcal{G} \cdot n := \mathcal{P} \cdot [n] \setminus \emptyset$;
- ☐ for any $S \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $S_1, \dots, S_n \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$,

$$\gamma_{n,m} \cdot S \cdot S_1 \cdot \; \cdots \; \cdot S_n := \bigcup_{i \in S} S_i;$$

- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n} := \{i\}$.

We have for instance

$$\gamma_{5,9} \cdot \{1,3,4\} \cdot \{2\} \cdot \{6,8\} \cdot \{7,8\} \cdot \{2,3\} \cdot \{1,2,3,4\} = \{2\} \cup \{7,8\} \cup \{2,3\} = \{2,3,7,8\}.$$

The $2,5$-projection is $\{2\}$.

There are in SL some nontrivial identities. For instance, for any $S_1, S_2 \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$,

$$\gamma_{2,m} \cdot \{1,2\} \cdot S_1 \cdot S_2 = S_1 \cup S_2 = S_2 \cup S_1 = \gamma_{2,m} \cdot \{1,2\} \cdot S_2 \cdot S_1.$$

The *trivial clone* is the unique clone having its underlying set $\mathcal{G}$ satisfying that $\mathcal{G}\cdot n$ is a singleton for any $n \in \mathbb{N}$.

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m\in\mathbb{N}}, (\mathbb{1}_{i,n})_{n\in\mathbb{N}, i\in[n]}\right)$ and $\mathcal{C}' := \left(\mathcal{G}', (\gamma'_{n,m})_{n,m\in\mathbb{N}}, (\mathbb{1}'_{i,n})_{n\in\mathbb{N}, i\in[n]}\right)$ be two clones.

A function $\phi : \mathcal{G} \to \mathcal{G}'$ is a *clone morphism* if

- ☐ for any $x \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, $\phi\cdot x \in \mathcal{G}'\cdot n$;
- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\phi\cdot\mathbb{1}_{i,n} = \mathbb{1}'_{i,n}$;
- ☐ for any $x \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, and $y_1, \dots, y_n \in \mathcal{G}\cdot m$, $m \in \mathbb{N}$,

$$\phi\cdot\underbracket{\gamma_{n,m}\cdot x\cdot y_1\cdot \cdots \cdot y_n} = \gamma'_{n,m}\cdot\underbracket{\phi\cdot x}\cdot\underbracket{\phi\cdot y_1}\cdot \cdots \cdot\underbracket{\phi\cdot y_n}.$$

The clone $\mathcal{C}'$ is a *subclone* of $\mathcal{C}$ if

- ☐ $\mathcal{G}'$ is a sub-graded set of $\mathcal{G}$;
- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}'_{i,n} = \mathbb{1}_{i,n}$;
- ☐ for any $n, m \in \mathbb{N}$, $x \in \mathcal{G}'\cdot n$, and $y_1, \dots, y_n \in \mathcal{G}'\cdot m$,

$$\gamma'_{n,m}\cdot x\cdot y_1\cdot \cdots \cdot y_n = \gamma_{n,m}\cdot x\cdot y_1\cdot \cdots \cdot y_n.$$

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ be a clone.

Given a set X of elements of the underlying set of $\mathcal{G}$, the *subclone of $\mathcal{C}$ generated by X* is the smallest (w.r.t. inclusion of the underlying sets of the underlying graded sets) subclone $\mathcal{C}^{(X)}$ of $\mathcal{C}$ such that the underlying set of the underlying graded set of $\mathcal{C}^{(X)}$ contains X.

When X is such that $\mathcal{C}^{(X)} = \mathcal{C}$, X is a *generating set* of $\mathcal{C}$.

When X is a generating set of $\mathcal{C}$ and for any $X' \subseteq X$, $\mathcal{C}^{(X')} = \mathcal{C}^{(X)}$ implies $X' = X$, X is a *minimal generating set* of $\mathcal{C}$.

Example

It is possible to show that $\{g\}$, where $g := \{1, 2\}$ is an element of arity 2 of the underlying set of the underlying graded set of SL, is a minimal generating set of SL. For instance,

$$\{1, 3, 4\} = \gamma_{2,4} \cdot g \cdot \mathbb{1}_{1,4} \cdot \underbracket{\gamma_{2,4} \cdot g \cdot \mathbb{1}_{3,4} \cdot \mathbb{1}_{4,4}}.$$

A *clone congruence* on $\mathcal{C}$ is an equivalence relation $\equiv$ on the underlying set of $\mathcal{G}$ such that

- ☐ for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $[x]_{\equiv} \subseteq \mathcal{G} \cdot n$;
- ☐ for any $x, x' \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $y_1, y_1', \dots, y_n, y_n' \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$, if $x \equiv x'$ and $y_i \equiv y_i'$ for all $i \in [n]$, then

$$\gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n \equiv \gamma_{n,m} \cdot x' \cdot y_1' \cdot \cdots \cdot y_n'.$$

The *quotient* of $\mathcal{C}$ by $\equiv$ is the clone $\mathcal{C}/\equiv := \left(\mathcal{G}/\equiv, \left(\gamma'_{n,m}\right)_{n,m \in \mathbb{N}}, \left(\mathbb{1}'_{i,n}\right)_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- ☐ $\mathcal{G}/\equiv$ is the graded set satisfying, for any $n \in \mathbb{N}$, $\mathcal{G}/\equiv \cdot n = \{[x]_{\equiv} : x \in \mathcal{G} \cdot n\}$;
- ☐ for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $y_1, \dots, y_n \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$,

$$\gamma'_{n,m} \cdot [x]_{\equiv} \cdot [y_1]_{\equiv} \cdot \cdots \cdot [y_n]_{\equiv} = [\gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n]_{\equiv};$$

- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}'_{i,n} = [\mathbb{1}_{i,n}]_{\equiv}$.

/ Clones and varieties

10.2. Clones of pigmented words

Let X be a set. An *X-pigmented letter* is a pair (i, x) such that $i \in \mathbb{N} \setminus \{0\}$ and $x \in X$. This pair is denoted by i^x. The *value* (resp. *pigment*) of i^x is i (resp. x).

A finite sequence of X-pigmented letters is an *X-pigmented word*.

Let $\mathcal{M}$ be a **monoid** with product $\star$ and unit e.

Let the triple $\mathbf{P}{\cdot}\mathcal{M} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

☐ $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}{\cdot}n$ is the set of $\mathcal{M}$-pigmented words whose values of letters range all between 1 and n;

☐ for any $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathcal{G}{\cdot}n$, $n \in \mathbb{N}$, $\ell \in \mathbb{N}$, $w_1, \dots, w_n \in \mathcal{G}{\cdot}m$, $m \in \mathbb{N}$,

$$\gamma_{n,m}{\cdot}\underbrace{i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}}{\cdot}w_1{\cdot} \cdots {\cdot}w_n := (\alpha_1 \mathbin{\bar{\star}} w_{i_1}) \blacksquare \dots \blacksquare (\alpha_\ell \mathbin{\bar{\star}} w_{i_\ell})$$

where for any $\beta \in \mathcal{M}$ and any $j_1^{\beta_1} \dots j_k^{\beta_k} \in \mathcal{G}$, $k \in \mathbb{N}$,

$$\beta \mathbin{\bar{\star}} j_1^{\beta_1} \dots j_k^{\beta_k} := j_1^{\beta \star \beta_1} \dots j_k^{\beta \star \beta_k};$$

☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n} := i^{\mathrm{e}}$.

Theorem [Clones of $\mathcal{M}$-pigmented words]

For any monoid $\mathcal{M}$, $\mathbf{P}\cdot\mathcal{M}$ is a clone.

This result comes from [S. Giraudo, Clones of pigmented words and realizations of special classes of monoids, 2026].

Examples

Let $\mathcal{M}_{\mathrm{w}}$ be the free monoid $(\{a, b\}^*, \centerdot, \epsilon)$.

The $3, 8$-projection of $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$ is $\mathbb{1}_{3,8} = 3^{\epsilon}$ and the $3, 4$-projection of $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$ is $\mathbb{1}_{3,4} = 3^{\epsilon}$. Even if they are denoted through the same $\mathcal{M}_{\mathrm{w}}$-pigmented word 3^{ϵ}, $\mathbb{1}_{3,8}$ and $\mathbb{1}_{3,4}$ are **different** elements. This remark applies also for other clones: for instance, in SL, there is a copy of the element $\{1, 3, 4\}$ for any arity $n \geqslant 4$.

Moreover, in $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$, we have

$$\gamma_{4,3}\cdot 2^{ab}4^{a}1^{\epsilon}1^{ba} \;\cdot 2^{a}2^{\epsilon} \;\cdot 3^{ba}1^{a}3^{b} \;\cdot 1^{a} \;\cdot \epsilon \;=\; 3^{ab\centerdot ba}1^{ab\centerdot a}3^{ab\centerdot b} \centerdot \epsilon \centerdot 2^{\epsilon\centerdot a}2^{\epsilon\centerdot\epsilon} \centerdot 2^{ba\centerdot a}2^{ba\centerdot\epsilon} \;=\; 3^{abba}1^{aba}3^{abb}2^{a}2^{\epsilon}2^{baa}2^{ba}.$$

Let $\mathcal{M}$ be a monoid.

Let $\equiv_{\mathrm{sort}}$ be the equivalence relation on the underlying set of the underlying graded set of $\mathbf{P}\cdot\mathcal{M}$ defined by $w \equiv_{\mathrm{sort}} w'$ if, for any $\mathcal{M}$-pigmented letter i^{α}, w and w' have both the same number of occurrences of i^{α}.

Example

The $\mathcal{M}_{\mathrm{w}}$-pigmented words $2^{ab}3^{ba}2^{b}$ and $2^{b}2^{ab}3^{ba}$ of arity 3 of $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$ are $\equiv_{\mathrm{sort}}$-equivalent.

Proposition [Sorting congruence on $\mathbf{P}\cdot\mathcal{M}$]

For any monoid $\mathcal{M}$, the equivalence relation $\equiv_{\mathrm{sort}}$ is a clone congruence on $\mathbf{P}\cdot\mathcal{M}$.

Example

In the quotient $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}/\equiv_{\mathrm{sort}}$, we have

$$\gamma_{2,4}\cdot\left[1^{a}2^{b}2^{ab}\right]_{\equiv_{\mathrm{sort}}}\cdot\left[1^{aa}3^{ab}\right]_{\equiv_{\mathrm{sort}}}\cdot\left[1^{\epsilon}1^{a}1^{aa}\right]_{\equiv_{\mathrm{sort}}} = \left[1^{aaa}1^{ab}1^{aba}1^{abaa}1^{b}1^{ba}1^{baa}3^{aab}\right]_{\equiv_{\mathrm{sort}}}.$$

Let $\mathcal{M}$ be a monoid.

Let, for any $k \in \mathbb{N}$, first_k be the function sending an $\mathcal{M}$-pigmented word w to the subword of w obtained by deleting all occurrences of $\mathcal{M}$-pigmented letters having $i \in \mathbb{N} \setminus \{0\}$ as value when there are k or more occurrences of $\mathcal{M}$-pigmented letters of value i on the left.

Let also $\equiv_{\mathrm{first}_k}$ be the kernel of first_k.

Example

For any $\alpha_1, \dots, \alpha_9 \in \mathcal{M}$,

$$\mathrm{first}_2 \cdot 1^{\alpha_1} 2^{\alpha_2} 1^{\alpha_3} 3^{\alpha_4} 1^{\alpha_5} 3^{\alpha_6} 2^{\alpha_7} 4^{\alpha_8} 3^{\alpha_9} = 1^{\alpha_1} 2^{\alpha_2} 1^{\alpha_3} 3^{\alpha_4} 3^{\alpha_6} 2^{\alpha_7} 4^{\alpha_8}.$$

Proposition [First congruence on $\mathbf{P} \cdot \mathcal{M}$]

For any monoid $\mathcal{M}$ and $k \in \mathbb{N}$, the equivalence relation $\equiv_{\mathrm{first}_k}$ is a clone congruence on $\mathbf{P} \cdot \mathcal{M}$.

Example

In the quotient $\mathbf{P} \cdot \mathcal{M}_{\mathrm{w}} / \equiv_{\mathrm{first}_1}$, we have for instance

$$\gamma_{3,3} \cdot \left[2^{ba} 1^{bab} 3^{b}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[2^{b} 1^{a}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[2^{aa} 3^{a}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[3^{aa} 1^{a} 2^{\epsilon}\right]_{\equiv_{\mathrm{first}_1}} = \left[2^{baaa} 3^{baa} 1^{baba}\right]_{\equiv_{\mathrm{first}_1}}.$$

When $\mathcal{M}$ is the **trivial monoid** $\mathbf{E} := (\{\mathrm{e}\}, \star, \mathrm{e})$, the elements of the underlying set of the underlying graded set of $\mathbf{P}{\cdot}\mathbf{E}$ are *monochrome pigmented words*. In this case, we denote simply by $i_1 \dots i_\ell$, $\ell \in \mathbb{N}$, the monochrome pigmented word $i_1^{\mathrm{e}} \dots i_\ell^{\mathrm{e}}$.

Examples

In $\mathbf{P}{\cdot}\mathbf{E}$, we have

$$\gamma_{4,3} \cdot 2411 \;\cdot 22 \;\cdot 313 \;\cdot 1 \;\cdot \epsilon \;=\; 313.\epsilon.22.22 \;=\; 3132222.$$

In $\mathbf{P}{\cdot}\mathbf{E}/\equiv_{\mathrm{sort}}$, we have

$$\gamma_{2,4} \cdot [122]_{\equiv_{\mathrm{sort}}} \cdot [13]_{\equiv_{\mathrm{sort}}} \cdot [111]_{\equiv_{\mathrm{sort}}} = [11111113]_{\equiv_{\mathrm{sort}}}.$$

In $\mathbf{P}{\cdot}\mathbf{E}/\equiv_{\mathrm{first}_1}$, we have

$$\gamma_{3,3} \cdot [213]_{\equiv_{\mathrm{first}_1}} \cdot [21]_{\equiv_{\mathrm{first}_1}} \cdot [23]_{\equiv_{\mathrm{first}_1}} \cdot [312]_{\equiv_{\mathrm{first}_1}} = [231]_{\equiv_{\mathrm{first}_1}}.$$

It is possible to build a **hierarchy of clones** by considering quotients $\mathbf{P}{\cdot}\mathcal{M}$, where $\mathcal{M}$ is a monoid, by using the clone congruences $\equiv_{\mathrm{sort}}$ and $\equiv_{\mathrm{first}_k}$, $k \in \mathbb{N}$, their intersections, and their reversions.

Here is the diagram, where arrows are surjective clone morphisms (without further details):

$$\begin{tikzcd}
 & \mathbf{P}(\mathcal{M}) \arrow[d, two heads] & \\
 & \mathsf{Pill}_{k,k'}(\mathcal{M}) \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, two heads] & \\
\mathsf{Stal}_k(\mathcal{M}) \arrow[d, two heads] \arrow[dr, two heads] & \mathsf{Magn}_{k,k'}(\mathcal{M}) \arrow[dl, two heads] \arrow[dr, two heads] & \mathsf{Stal}^{\mathrm{r}}_{k'}(\mathcal{M}) \arrow[dl, two heads] \arrow[d, two heads] \\
\mathsf{Arra}_k(\mathcal{M}) \arrow[dr, two heads] & \mathsf{WInc}(\mathcal{M}) \arrow[d, two heads] & \mathsf{Arra}^{\mathrm{r}}_{k'}(\mathcal{M}) \arrow[dl, two heads] \\
 & \mathsf{Inc}_{\min\{k,k'\}} \arrow[d, two heads] & \\
 & \mathsf{Inc}_0 &
\end{tikzcd}$$