10.3. Free clones and presentations

For any signature $\mathcal{S}$, let the triple $\mathcal{FC}\cdot\mathcal{S} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m\in\mathbb{N}}, (\mathbb{1}_{i,n})_{n\in\mathbb{N}, i\in[n]}\right)$ such that

- $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}\cdot n$ is the set of labeled $\mathcal{S}$-terms $\mathfrak{t}$ such that $\mathrm{rk_v}\cdot\mathfrak{t} \leqslant n$;
- for any $\mathfrak{t} \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathcal{G}\cdot m$, $m \in \mathbb{N}$,

$$\gamma_{n,m}\cdot\mathfrak{t}\cdot\mathfrak{t}_1\cdot \cdots \cdot\mathfrak{t}_n := \mathfrak{t}[\mathfrak{t}_1, \dots, \mathfrak{t}_n];$$

- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is the labeled $\mathcal{S}$-term consisting in one leaf decorated by i.

Let $\iota : \mathcal{S} \to \mathfrak{T}\cdot\mathcal{S}\cdot\underline{\mathbb{N} \setminus \{0\}}$ be the function such that for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, $\iota\cdot c$ is the labeled $\mathcal{S}$-term $c1\dots n$.

Theorem [Free clones]

For any signature $\mathcal{S}$, $\mathcal{FC}\cdot\mathcal{S}$ is a clone. Moreover, for any clone $\mathcal{C}$ and any rank-preserving function f from the underlying set of $\mathcal{S}$ to the underlying set of the underlying graded set of $\mathcal{C}$, there exists a unique clone morphism ϕ from $\mathcal{FC}\cdot\mathcal{S}$ to $\mathcal{C}$ such that $f = \phi \circ \iota$.

The class of clones together with clone morphisms forms a category. Theorem [Free clones] says that for any signature $\mathcal{S}$, $\mathcal{FC}\cdot\mathcal{S}$ is a free object in this category.

Let $\mathcal{C}$ be a clone.

A *presentation* of $\mathcal{C}$ is an equational presentation $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ such that $\mathcal{C}$ is isomorphic to $\mathcal{FC}\cdot\mathcal{S}/\equiv$ where $\equiv$ is the smallest clone congruence on $\mathcal{FC}\cdot\mathcal{S}$ containing $\sim$.

Example

The clone SL admits the equational presentation $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ as presentation, where $\mathcal{S}$ is the signature containing a binary constant $\wedge$ and $\sim$ is defined by $\wedge\llcorner\underline{\wedge 12}\lrcorner 3 \quad \sim \quad \wedge 1\llcorner\underline{\wedge 23}\lrcorner$, $\wedge 12 \quad \sim \quad \wedge 21$, and $\wedge 11 \sim 1$.

A clone $\mathcal{C}$ can have different presentations, even on non-isomorphic signatures.

Example

Let the equational presentation $(\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ is the signature containing a binary constant $\wedge$ and a ternary constant t, and $\sim'$ is defined by $\wedge\llcorner\underline{\wedge 12}\lrcorner 3 \sim' \wedge 1\llcorner\underline{\wedge 23}\lrcorner$, $\wedge 12 \sim' \wedge 21$, $\wedge 11 \sim' 1$, and $t123 \sim' \wedge\llcorner\underline{\wedge 12}\lrcorner 3$.

Then, $\mathcal{E}'$ is another presentation of SL.

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ be a clone.

For any $n \in \mathbb{N}$, the *$\mathcal{C}, n$-evaluation function* is the function defined, for any $\mathcal{G}, [n]$-term $\mathfrak{t}$, by

$$\mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t} := \begin{cases} \mathbb{1}_{i,n} & \text{if } \mathfrak{t} = i \text{ with } i \in [n], \\ \gamma_{m,n} \cdot c \cdot \underbrace{\mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t}_1} \cdot \cdots \cdot \underbrace{\mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t}_m} & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \dots \mathfrak{t}_m, c \in \mathcal{G} \cdot m, m \in \mathbb{N}, \mathfrak{t}_i \in \mathfrak{T} \cdot \mathcal{G} \cdot [n], i \in [m]. \end{cases}$$

Example

In SL, we have

$$\mathrm{ev}_{\mathrm{SL},4} \cdot \{1,3\} \underbrace{\{2,4\} 1324} 14 = \{3,4\}.$$

Proposition [Raw presentations of clones]

Let $\mathcal{C}$ be a clone having $\mathcal{G}$ as underlying graded set. The equational presentation $(\mathcal{G}, \mathbb{N} \setminus \{0\}, \sim)$ is a presentation of $\mathcal{C}$, where $\sim$ is defined by $\mathfrak{t} \sim \mathfrak{t}'$ if $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{G} \cdot [n]$, $n \in \mathbb{N}$, and $(\mathfrak{t}, \mathfrak{t}') \in \mathrm{Ker} \cdot \mathrm{ev}_{\mathcal{C},n}$.

The presentation of $\mathcal{C}$ described by Proposition [Raw presentations of clones] is the *raw presentation* of $\mathcal{C}$.

10.4. Algebras over clones

For any set A, let the triple $\mathrm{End}\cdot A := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- ☐ $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}\cdot n$ is the set of n-operations on A;
- ☐ for any $\phi \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, $\psi_1, \dots, \psi_n \in \mathcal{G}\cdot m$, $m \in \mathbb{N}$, and $a_1, \dots, a_m \in A$,
$$\gamma_{n,m}\cdot\phi\cdot\psi_1\cdot \dots \cdot\psi_n\cdot a_1\cdot \dots \cdot a_m := \phi\cdot\lfloor\psi_1\cdot a_1\cdot \dots \cdot a_m\rfloor\cdot \dots \cdot\lfloor\psi_n\cdot a_1\cdot \dots \cdot a_m\rfloor;$$
- ☐ for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is the n-operation on A defined, for any $a_1, \dots, a_n \in A$, by
$$\mathbb{1}_{i,n}\cdot a_1\cdot \dots \cdot a_n := a_i.$$

Proposition [Clones of endomorphisms]

For any set A, $\mathrm{End}\cdot A$ is a clone.

The clone $\mathrm{End}\cdot A$ is the *clone of endomorphisms on* A.

Exercise ○○○○○

Prove Proposition [Clones of endomorphisms].

Let $\mathcal{C}$ be a clone and A be a set.

An *algebra over $\mathcal{C}$ on A* is a $\mathcal{G}$-algebra $(A, \mathcal{G}, \mathrm{op})$ such that

- ☐ $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$;
- ☐ there exists a clone morphism ϕ from $\mathcal{C}$ to $\mathrm{End} \cdot A$ such that for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $\mathrm{op} \cdot x$ is the n-operation $\phi \cdot x$ on A.

Observe that if X is a generating set of $\mathcal{C}$, then ϕ is uniquely determined by the images by ϕ on X. Hence, to define an algebra over $\mathcal{C}$, it is enough to describe $\mathrm{op} \cdot x$ for any $x \in X$.

Example

Consider the clone SL and the set $A := \mathbb{Z}$.
Let ϕ be the clone morphism from SL to $\mathrm{End} \cdot A$ defined, for any $S \in \mathcal{P} \cdot [n] \setminus \emptyset$, $n \in \mathbb{N}$, by

$$\phi \cdot S \cdot a_1 \cdot \cdots \cdot a_n := \max\{a_i : i \in S\}.$$

This defines an algebra $(A, \mathcal{G}, \mathrm{op})$ over SL on A where $\mathcal{G}$ is the underlying graded set of SL. We have for instance

$$\mathrm{op} \cdot \{1, 3\} \cdot -4 \cdot 3 \cdot 2 = \max\{-4, 2\} = 2.$$

Since $\{\{1, 2\}\}$ is a generating set of SL, the operations $\mathrm{op} \cdot S$ are uniquely determined for any $S \in \mathcal{P} \cdot [n] \setminus \emptyset$, $n \in \mathbb{N}$, by $\mathrm{op} \cdot \{1, 2\} \cdot a_1 \cdot a_2 = \max\{a_1, a_2\}$, where $a_1, a_2 \in A$.

Theorem [Algebras over clones and varieties]

Let $\mathcal{C}$ be a clone having $\mathcal{G}$ as underlying graded set.

1. The class of algebras over $\mathcal{C}$ is a variety of $\mathcal{G}$-algebras.
2. This variety of $\mathcal{C}$-algebras is the variety of the raw presentation of $\mathcal{C}$.

Theorem [Algebras over clones and varieties] relies on Theorem [Birkhoff's Variety Theorem] and says that clones, through their algebras, describe varieties of algebras.

Example

Consider the algebras over the clone SL.

Since the set $\{g\}$, where $g := \{1, 2\}$, is a minimal generating set of SL, to define an algebra $\mathcal{A} := (A, \mathcal{G}, \mathrm{op})$ over SL where A is a set and $\mathcal{G}$ is the underlying graded set of SL, it is enough to define $\mathrm{op} \cdot g$.

We can check that, due to the definition of SL, $\mathrm{op} \cdot g$ is an associative, commutative, and idempotent 2-operation on A. Therefore, $\mathcal{A}$ is a semilattice.

Definition

Let $\mathcal{E}$ and $\mathcal{E}'$ be two equational presentations having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables. If there exists a clone $\mathcal{C}$ which admits both $\mathcal{E}$ and $\mathcal{E}'$ as presentations, then $\mathcal{E}$ and $\mathcal{E}'$ are *equivalent*.

Definition

Let $\mathcal{V}$ and $\mathcal{V}'$ be two varieties. If there exist two equivalent equational presentations $\mathcal{E}$ and $\mathcal{E}'$ such that the class of algebras over $\mathcal{E}$ (resp. $\mathcal{E}'$) is $\mathcal{V}$ (resp. $\mathcal{V}'$), then $\mathcal{V}$ and $\mathcal{V}'$ are *equivalent*.

Thus, clones encode equivalence classes of varieties through their algebras.

A natural question about a variety $\mathcal{V}$ consists in exhibiting a variety $\mathcal{V}'$ such that $\mathcal{V}$ and $\mathcal{V}'$ are equivalent and $\mathcal{V}'$ is the simplest possible one. By ‘‘simplest’’, we mean a variety which is described by an equational presentation $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ such that

- ☐ $\mathcal{S}$ is the smallest possible, for a certain size notion on signatures;
- ☐ and/or $\sim$ is the smallest possible, for a certain size notion on elementary identity relations.

Exercise ○○○○○

Prove that the equational presentations **Groups** and **PHeaps** are equivalent.

Exercise ○○○○○

Let $\mathbf{AG_1} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be the usual equational presentation of *abelian groups*, where $\mathcal{S}$ contains a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \llcorner m12 \lrcorner 3 \sim m1 \llcorner m23 \lrcorner$, $me1 \sim 1$, $m \llcorner i1 \lrcorner 1 \sim e$, and $m12 \sim m21$.

Let also the equational presentation $\mathbf{AG_2} := (\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ contains one nullary constant c and one binary constant d, and $\sim'$ is defined by $d11 \sim' c$ and $d1 \llcorner d2 \llcorner d3 \llcorner d12 \lrcorner \lrcorner \lrcorner \sim' 3$.

Prove that $\mathbf{AG_1}$ and $\mathbf{AG_2}$ are equivalent.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two clones, and ϕ be a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$.

Let $\mathcal{A}' := (A, \mathcal{G}', \mathrm{op}')$ be an algebra over $\mathcal{C}'$ on a set A. Let $\phi \cdot \mathcal{A}' := (A, \mathcal{G}, \mathrm{op})$ be the triple such that

- $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$;
- for any $g \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $\mathrm{op} \cdot g$ is the n-operation on A defined, for any $a_1, \dots, a_n \in A$, by

$$\mathrm{op} \cdot g \cdot a_1 \cdot \cdots \cdot a_n := \mathrm{op}' \cdot \llcorner \phi \cdot g \lrcorner \cdot a_1 \cdot \cdots \cdot a_n.$$

Theorem [Clone morphisms and algebras]

Let $\mathcal{C}$ and $\mathcal{C}'$ be two clones, and ϕ be a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$. If $\mathcal{A}'$ is an algebra over $\mathcal{C}'$, then $\phi \cdot \mathcal{A}'$ is an algebra over $\mathcal{C}$.

Theorem [Clone morphisms and algebras] says that a clone morphism from a clone $\mathcal{C}$ to a clone $\mathcal{C}'$ gives rise to a transformation of an algebra over $\mathcal{C}'$ to an algebra over $\mathcal{C}$. This transformation can be described as a functor from the category of algebras over $\mathcal{C}'$ to the category of algebras over $\mathcal{C}$.

Example

Let the equational presentation $\mathcal{E} := (\mathrm{MagC}, \mathbb{N} \setminus \{0\}, \sim)$ where $\sim$ satisfies $m\underline{m12}3 \sim m1\underline{m23}$, $m\underline{m12}1 \sim m12$, $m1c \sim 1$, and $mc1 \sim 1$.

This is the equational presentation **LBMonoids** of *left-regular band monoids*.

Let $\mathcal{C} := \mathcal{FC}{\cdot}\mathrm{MagC}/\equiv$ where $\equiv$ is the smallest clone congruence containing $\sim$.

Let the equational presentation $\mathcal{E}' := (\mathrm{MagC}, \mathbb{N} \setminus \{0\}, \sim')$ where $\sim'$ satisfies $m\underline{m12}3 \sim' m1\underline{m23}$, $m1c \sim' 1$, $mc1 \sim' 1$, $m12 \sim' m21$, and $m11 \sim' 1$.

This is the equational presentation **BSLattices** of bounded semilattices.

Let $\mathcal{C}' := \mathcal{FC}{\cdot}\mathrm{MagC}/{\equiv'}$ where $\equiv'$ is the smallest clone congruence containing $\sim'$.

Let ϕ be the function from the underlying set of the underlying graded set of $\mathcal{C}$ to the underlying set of the underlying graded set of $\mathcal{C}'$ sending $[m]_{\equiv}$ to $[m]_{\equiv'}$ and $[c]_{\equiv}$ to $[c]_{\equiv'}$. It is possible to show that ϕ can be uniquely extended as a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$.

By Theorem [Clone morphisms and algebras], if $\mathcal{A}$ is a bounded semilattice, then $\mathcal{A}$ is also a left-regular band monoid.

10.5. Clone realizations of varieties

Let $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be an equational presentation.

A clone $\mathcal{C}$ admitting $\mathcal{E}$ as presentation is a *clone realization* of $\mathcal{E}$.

Let $\equiv$ be the smallest clone congruence on $\mathcal{FC}{\cdot}\mathcal{S}$ containing $\sim$. Fix a clone isomorphism $\bar{\phi}$ from $\mathcal{FC}{\cdot}\mathcal{S}/\equiv$ to $\mathcal{C}$, and set $\phi := \bar{\phi} \circ \pi \circ \iota$ where π is the canonical projection function from $\mathcal{FC}{\cdot}\mathcal{S}$ to $\mathcal{FC}{\cdot}\mathcal{S}/\equiv$.

The function ϕ is an *interpretation function* of $\mathcal{E}$ in $\mathcal{C}$.

For any $n \in \mathbb{N}$, let $\mathrm{ev}^{\phi}_{\mathcal{C},n}$ be the function defined, for any $\mathcal{S}, [n]$-term $\mathfrak{t}$, by $\mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t} := \mathrm{ev}_{\mathcal{C},n}{\cdot}\mathfrak{s}$, where $\mathfrak{s}$ is the $\mathcal{G}, [n]$-term obtained from $\mathfrak{t}$ by replacing each constant decoration c by $\phi{\cdot}c$, and $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$.

Theorem [Clone realizations and word problems]

Let $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be an equational presentation, $\mathcal{C}$ be a clone realization of $\mathcal{E}$, and ϕ be an interpretation function of $\mathcal{E}$ in $\mathcal{C}$. Then, for any $n \in \mathbb{N}$ and any $\mathcal{S}, [n]$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$ iff $\mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t} = \mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t}'$.

Example

Let the clone $\mathsf{Arra}_1 := \mathbf{P}\cdot\mathbf{E} / \equiv_{\mathrm{first}_1}$.

Let us identify each $\equiv_{\mathrm{first}_1}$-class $[w]_{\equiv_{\mathrm{first}_1}}$ of monochrome pigmented words by the unique monochrome pigmented word of minimal length in $[w]_{\equiv_{\mathrm{first}_1}}$. For instance, we identify $[2312]_{\equiv_{\mathrm{first}_1}}$ with 231 since $[2312]_{\equiv_{\mathrm{first}_1}} = \{231, 2231, 2331, 2311, 2311, 2312, \dots\}$. Under this identification, the underlying graded set $\mathcal{G}$ of Arra_1 is such that $\mathcal{G}\cdot n$, $n \in \mathbb{N}$, is the set of word on $[n]$ having at most one occurrence of each $i \in [n]$.

We have for instance

$$\gamma_{5,6}\cdot 3152\cdot 213\cdot 6132\cdot 3215\cdot 2\cdot 6512 = 32156.$$

It is possible to show that Arra_1 is a clone realization of the equational presentation $\mathbf{LBMonoids}$ of left-regular band monoids. Let us assume this property.

Now, let ϕ be the function such that $\phi\cdot m = 12$ and $\phi\cdot c = \epsilon$. This is an interpretation function of $\mathbf{LBMonoids}$ in Arra_1.

Let the $\mathrm{MagC}, \mathbb{N} \setminus \{0\}$-terms $\mathfrak{t} := m \llcorner m2 \llcorner m21 \lrcorner \lrcorner \llcorner m \llcorner m2c \lrcorner 3 \lrcorner$ and $\mathfrak{t}' := m \llcorner m22 \lrcorner \llcorner m13 \lrcorner$. Since

$$\mathrm{ev}^{\phi}_{\mathsf{Arra}_1,3}\cdot\mathfrak{t} = \mathrm{ev}_{\mathsf{Arra}_1,3}\cdot \llcorner 12 \llcorner 12\,2 \llcorner 12\,2\,1 \lrcorner \lrcorner \llcorner 12 \llcorner 12\,2\,\epsilon \lrcorner 3 \lrcorner \lrcorner = 213$$

and

$$\mathrm{ev}^{\phi}_{\mathsf{Arra}_1,3}\cdot\mathfrak{t}' = \mathrm{ev}_{\mathsf{Arra}_1,3}\cdot \llcorner 12 \llcorner 1222 \lrcorner \llcorner 1213 \lrcorner \lrcorner = 213,$$

this shows that $\mathfrak{t} \equiv_{\mathbf{LBMonoids}} \mathfrak{t}'$.

Exercise

Let $\mathcal{M}$ be a monoid with product $\star$ and unit e.

Let $\mathcal{E}_{\mathcal{M}} := (\mathcal{S}_{\mathcal{M}}, \mathbb{N} \setminus \{0\}, \sim_{\mathcal{M}})$ be the equational presentation of *$\mathcal{M}$-pigmented monoids* where $\mathcal{S}_{\mathcal{M}}$ is the signature containing a nullary constant u, unary constants p_α, $\alpha \in \mathcal{M}$, and a binary constant $\star$, and $\sim_{\mathcal{M}}$ is defined by

$$\star \lfloor \star 1 2 \rfloor 3 \sim_{\mathcal{M}} \star 1 \lfloor \star 2 3 \rfloor,$$

$$\star \mathrm{u} 1 \sim_{\mathcal{M}} 1, \qquad \star 1 \mathrm{u} \sim_{\mathcal{M}} 1,$$

$$\mathrm{p}_\alpha \lfloor \star 1 2 \rfloor \sim_{\mathcal{M}} \star \lfloor \mathrm{p}_\alpha 1 \rfloor \lfloor \mathrm{p}_\alpha 2 \rfloor,$$

$$\mathrm{p}_\alpha \mathrm{u} \sim_{\mathcal{M}} \mathrm{u},$$

$$\mathrm{p}_{\alpha_1} \lfloor \mathrm{p}_{\alpha_2} 1 \rfloor \sim_{\mathcal{M}} \mathrm{p}_{\alpha_1 \star \alpha_2} 1,$$

$$\mathrm{p}_{\mathrm{e}} 1 \sim_{\mathcal{M}} 1,$$

for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

Show that $\mathbf{P} \cdot \mathcal{M}$ is a clone realization of $\mathcal{E}_{\mathcal{M}}$.

Clones and varieties

10.6. Tietze rules

The *Tietze rules* consist in the following four rules allowing us to transform an equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ into another one:

1. [Constant Adding] Transform $\mathcal{E}$ into $(\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ such that there exists a new constant c and $n \in \mathbb{N}$ such that $\mathcal{S}' \cdot n = \mathcal{S} \cdot n \sqcup \{c\}$, for any $m \in \mathbb{N} \setminus \{n\}$, $\mathcal{S}' \cdot m = \mathcal{S} \cdot m$, and $\sim' = \sim \cup \{(c\,1 \ldots n, \mathfrak{t})\}$ where $\mathfrak{t}$ is an $\mathcal{S}, [n]$-term;

2. [Constant Deleting] Transform $\mathcal{E}$ into $\mathcal{E}'$ if $\mathcal{E}$ is obtained from $\mathcal{E}'$ through an application of the Constant Adding rule;

3. [Elementary Identity Adding] Transform $\mathcal{E}$ into $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim')$ such that $\sim' = \sim \cup \{(\mathfrak{t}_1, \mathfrak{t}_2)\}$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathcal{S}, \mathbb{N} \setminus \{0\}$-terms satisfying $(\mathfrak{t}_1, \mathfrak{t}_2) \notin \sim$ and $\mathfrak{t}_1 \approx_{\mathcal{E}} \mathfrak{t}_2$;

4. [Elementary Identity Deleting] Transform $\mathcal{E}$ into $\mathcal{E}'$ if $\mathcal{E}$ is obtained from $\mathcal{E}'$ through an application of the Elementary Identity Adding rule.

Two equational presentations $\mathcal{E}$ and $\mathcal{E}'$ having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables are *Tietze equivalent* if it is possible to transform $\mathcal{E}$ into $\mathcal{E}'$ by means of the iterative application of the Tietze rules.

Example 1/6

Recall that **Groups** is the equational presentation **Groups** $:= (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ where $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \llcorner m12 \lrcorner 3 \sim m1 \llcorner m23 \lrcorner$, $me1 \sim 1$, $m1e \sim 1$, $m \llcorner i1 \lrcorner 1 \sim e$, and $m1 \llcorner i1 \lrcorner \sim e$.

Recall moreover that **PHeaps** is the equational presentation **PHeaps** $:= (\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ is the signature containing a nullary constant e' and a ternary constant p', and $\sim'$ is defined by $p'112 \sim' 2$, $p'122 \sim' 1$, and $p' \llcorner p'123 \lrcorner 45 \sim' p'12 \llcorner p'345 \lrcorner$.

Let us show that **PHeaps** and **Groups** are Tietze equivalent.

Set $P_0 :=$ **PHeaps**.

Starting from P_0, we apply three times the Constant Adding rule:

1. we add a new nullary constant e together with the elementary identity
$$e \sim_0 e';$$
2. we add a new unary constant i together with the elementary identity
$$i1 \sim_0 p'e1e;$$
3. we add a new binary constant m together with the elementary identity
$$m12 \sim_0 p'1e2.$$

Let P_1 be the obtained equational presentation and let $\equiv_1 := \approx_{P_1}$.

Example 2/6

In P_1, the five identities of **Groups** are derivable:

$$m \underline{m12} 3 \quad \equiv_1 \quad p' \underline{p'1e2} e3 \quad \equiv_1 \quad p'1e \underline{p'2e3} \quad \equiv_1 \quad m1 \underline{m23},$$

$$me1 \quad \equiv_1 \quad p'ee1 \quad \equiv_1 \quad 1,$$

$$m1e \quad \equiv_1 \quad p'1ee \quad \equiv_1 \quad 1,$$

$$m \underline{i1} 1 \quad \equiv_1 \quad p' \underline{p'e1e} e1 \quad \equiv_1 \quad p'e1 \underline{p'ee1} \quad \equiv_1 \quad p'e11 \quad \equiv_1 \quad e,$$

$$m1 \underline{i1} \quad \equiv_1 \quad p'1e \underline{p'e1e} \quad \equiv_1 \quad \underline{p'1ee} 1e \quad \equiv_1 \quad p'11e \quad \equiv_1 \quad e.$$

Hence, by five applications of the Elementary Identity Adding rule, we add the five elementary identities of **Groups**.

Let P_2 be the obtained equational presentation and let $\equiv_2 := \approx_{P_2}$.

Example 3/6

In P_2, we have

$$e' \equiv_2 e,$$

and

$$m\underline{m1\underline{i2}}3 \equiv_2 p'\underline{p'1e\underline{p'e2e}}e3 \equiv_2 p'1e\underline{p'\underline{p'e2e}e3} \equiv_2 p'1e\underline{p'e2\underline{p'ee3}}$$
$$\equiv_2 p'1e\underline{p'e23} \equiv_2 \underline{p'1ee}23 \equiv_2 p'123.$$

Hence, by two further applications of the Elementary Identity Adding rule, we add the two elementary identities

$$e' \sim_2 e$$

and

$$p'123 \sim_2 m\underline{m1\underline{i2}}3.$$

Let P_3 be the obtained equational presentation and let $\equiv_3 := \approx_{P_3}$.

Example 4/6

Starting from P_3, we now delete the three original identities of **PHeaps**.

This is possible because in P_3, by removing respectively these identities, we still have

$$p'112 \equiv_3 m\underline{m1\underline{i1}}2 \equiv_3 me2 \equiv_3 2,$$

$$p'122 \equiv_3 m\underline{m1\underline{i2}}2 \equiv_3 m1\underline{m\underline{i2}2} \equiv_3 m1e \equiv_3 1,$$

and

$$p'\underline{p'123}45 \equiv_3 m\underline{m\underline{m\underline{m1\underline{i2}}3}\underline{i4}}5 \equiv_3 m\underline{m\underline{m1\underline{i2}}3}\underline{m\underline{i4}5}$$

$$\equiv_3 m\underline{m1\underline{i2}}\underline{m3\underline{m\underline{i4}5}} \equiv_3 m\underline{m1\underline{i2}}\underline{m\underline{m3\underline{i4}}5} \equiv_3 p'12\underline{p'345}.$$

Therefore, by three applications of the Elementary Identity Deleting rule, we delete

$$p'112 \sim_3 2,$$

$$p'122 \sim_3 1,$$

and

$$p'\underline{p'123}45 \sim_3 p'12\underline{p'345}.$$

Let P_4 be the obtained equational presentation and let $\equiv_4 := \approx_{P_4}$.

Example 5/6

Starting from P_4, we now delete the three identities

$$e \sim_4 e',$$

$$i1 \sim_4 p'e1e,$$

and

$$m12 \sim_4 p'1e2.$$

This is possible because in P_4, by removing respectively these identities, we still have

$$e \quad \equiv_4 \quad e',$$

$$p'e1e \quad \equiv_4 \quad m \llcorner me \llcorner i1 \lrcorner \lrcorner e \quad \equiv_4 \quad m \llcorner i1 \lrcorner e \quad \equiv_4 \quad i1,$$

and, since

$$ie \quad \equiv_4 \quad m \llcorner ie \lrcorner e \quad \equiv_4 \quad e,$$

we have

$$p'1e2 \quad \equiv_4 \quad m \llcorner m1 \llcorner ie \lrcorner \lrcorner 2 \quad \equiv_4 \quad m \llcorner m1e \lrcorner 2 \quad \equiv_4 \quad m12.$$

Therefore, by three applications of the Elementary Identity Deleting rule, we delete these three identities.

Let P_5 be the obtained equational presentation.

Example 6/6

In P_5, the constant p' occurs only in

$$p'123 \sim_5 m \underbracket{m1\underbracket{i2}}3.$$

Hence, by the Constant Deleting rule, we delete p'.

In the obtained presentation, the constant e' occurs only in

$$e' \sim_5 e.$$

Hence, by the Constant Deleting rule, we delete e'.

The resulting equational presentation is exactly **Groups**.

Therefore, **PHeaps** and **Groups** are Tietze equivalent.

Proposition [Soundness of the Tietze rules]

If an equational presentation $\mathcal{E}'$ is obtained from an equational presentation $\mathcal{E}$ by one Tietze rule, then $\mathcal{E}$ and $\mathcal{E}'$ are equivalent.

An equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is *finite* if the underlying set of $\mathcal{S}$ is finite and $\sim$ is finite.

Theorem [Tietze rules and equivalence of equational presentations]

Two finite equational presentations $\mathcal{E}$ and $\mathcal{E}'$ having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables are equivalent iff $\mathcal{E}$ and $\mathcal{E}'$ are Tietze equivalent.

Examples

The equational presentations $\mathbf{Groups}$ and $\mathbf{PHeaps}$ are finite. Moreover, by the previous example, $\mathbf{Groups}$ and $\mathbf{PHeaps}$ are Tietze equivalent. Hence, by Theorem [Tietze rules and equivalence of equational presentations], $\mathbf{Groups}$ and $\mathbf{PHeaps}$ are equivalent.

Therefore, the variety of groups and the variety of pointed heaps are equivalent.

This shows that two varieties can be equivalent even when it appears that they are presented on different signatures.